The effect of a radial magnetic field on thermal convection in a rotating cylindrical annulus

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A b s t r a c t : The problem of rotating magnetoconvection in a cylindrical annulus in the presence of a radial magnetic field is considered. Previous studies show that convective instability has the form of waves travelling in azimuthal direction. Due to the applied magnetic field the dispersion curve for the Rayleigh number possesses two local minima. Here we explore this feature in dependence on the system parameters. We also find conditions for the two local minima existence.

Key words: rotating magnetoconvection, cylindrical annulus

1. Introduction

The principal convective structure inside planetary cores is believed to have a form of rolls directed along the rotation axis and propagating in the azimuthal direction (see Fig. 1). The model of a rotating cylindrical annulus has been set up by Busse (see e.g. *Busse et al., 1997* or *Schnaubelt and Busse, 1997*), as a prototype for theoretical modelling and experimental studies. Assuming the annular radius be large, planar approximation can be made which enables the use of Cartesian geometry. As a result, an infinite duct model (Fig. 2) is obtained.

Current study is based on the model considered by it Busse and Finocchi (1993), where homogeneous basic magnetic field has been imposed. Linear stability analysis has been performed for various magnetic field orientations. The convective instability has been found to be oscillatory, having the form of waves travelling in azimuthal direction. An interesting feature of the applied magnetic field is that dispersion curve for the Rayleigh number

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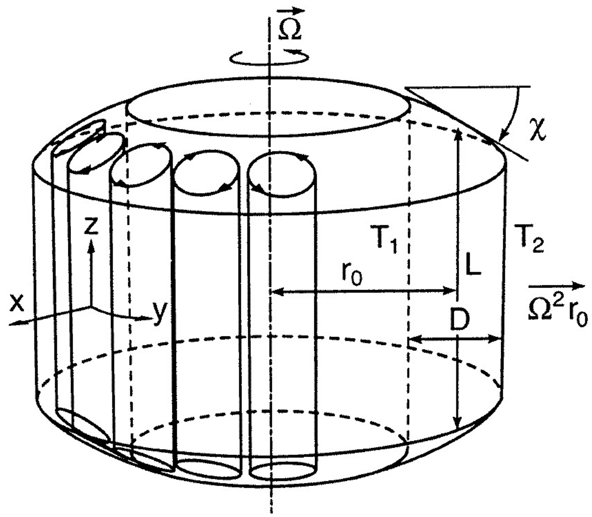

Fig. 1. Convection in a rotating annulus *(Busse et al., 1997)*.

exhibits two local minima. In later study by *Revallo and Ševčovič (2002)* the two minima phenomenon has been focused on. The condition for the mode resonance has been computed in dependence on the system parameters. As such, the local minima emerge at the same value of the Rayleigh number and can be identified as the two convective modes. Their linear and nonlinear stability has been explored.

In this paper we pursue the study by *Busse and Finocchi (1993)* for the special case of a radial basic magnetic field. We study the dispersion equation in a more detail for various system parameters choice. This study is focused on the question of the two mode formation and the condition for their existence. The paper is organised as follows. In Section 2 we describe the mathematical model according to *Busse and Finocchi (1993)*. In Section 3 we outline derivation of the dispersion equation and analyse its properties. Section 4 summarises the main results.

2. Mathematical formulation

Upon the local Cartesian approximation, the model considered is an infinite horizontal duct (Fig. 2), containing an electrically conducting Boussi-

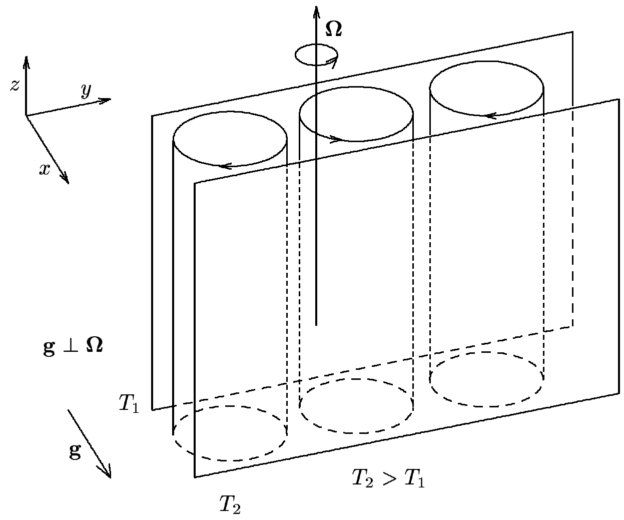

Fig. 2. Convection in a rotating duct *(Revallo and Ševčovič, 2002).*

nesq fluid. The duct rotates about the vertical axis and is permeated by a homogeneous horizontal magnetic field perpendicular to the sidewalls. Clearly, such configuration corresponds to the radial magnetic field in terms of the annulus model. The buoyancy is provided by the centrifugal force. The duct is exposed to the unstable temperature gradient which is directed opposite to the centrifugal force. The fluid is subjected to a convective instability occurring when heating measured by the Rayleigh number is strong enough. Convection in the underlying model can be described in terms of two scalar functions, the velocity potential ψ and temperature θ. We do not derive the mathematical formulation in this paper, for reference see *Busse and Finocchi (1993).*

The governing equations (those of *Busse and Finocchi, 1993*, Eqs. 6a, 6b) are as follows:

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\psi\,\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\psi\,\frac{\partial}{\partial y} - \Delta_2\right)\Delta_2\,\psi - \eta\,\frac{\partial}{\partial y}\psi + R\,\frac{\partial}{\partial y}\theta + Q\,\frac{\partial^2}{\partial x^2}\psi = 0\,, \tag{1}$$

$$P\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\psi\,\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\psi\,\frac{\partial}{\partial y}\right)\theta - \Delta_2\,\theta + \frac{\partial}{\partial y}\psi = 0\,, \tag{2}$$

where Δ_2 is the two dimensional Laplacian, $\Delta_2 = \partial^2_{xx} + \partial^2_{yy}$. The dimensionless parameters in the above equations are the Rayleigh number R, the Prandtl number P, the rotation parameter η and the Chandrasekhar number Q obeying the following definitions

$$R = \frac{\gamma\,(T_2 - T_1)\,\Omega^2\,r_0\,D^3}{\nu\kappa}\,, \quad P = \frac{\nu}{\kappa}\,, \quad \eta = \frac{4\,\Omega\,D^3\,\tan\chi}{\nu\,L}\,, \quad Q = \frac{B_0^2\,D^2}{\mu\,\rho\,\nu\,\lambda}\,. \tag{3}$$

Dimensional parameters in (3) are the angular velocity Ω, the basic applied magnetic field B_0, the temperature difference $T_2 - T_1$, the kinematic viscosity of the fluid ν, the thermal diffusivity κ, the magnetic diffusivity λ, the density ρ, the coefficient of thermal expansion γ and the magnetic permeability μ. The other dimensional parameters in (3) relate to the geometry of the original model in (Fig. 1), namely the height of the annulus L, the annular radius r_0, the thickness of the annular convective zone D and finally the angle χ measures inclination of the annular conical ends.

The sidewalls of the model are supposed to be stress-free and perfectly thermally conductive, i.e. in terms of the potentials we have

$$\psi(x,y,t) = \frac{\partial^2}{\partial x^2}\psi(x,y,t) = \theta(x,y,t) = 0 \qquad \text{at} \quad x = \pm\frac{1}{2}\,. \tag{4}$$

3. Properties of the dispersion equation

The linear stability problem will be considered in this study. A solution satisfying boundary conditions (4) can be sought in the form

$$\psi(x,y,t) = (Pi\omega + m^2\pi^2 + \alpha^2)\,\sin\left[m\pi(x+1/2)\right]\,\exp[i\alpha y + i\omega t]\,, \tag{5}$$

$$\theta(x,y,t) = (-i\alpha)\,\sin\left[m\pi(x+1/2)\right]\,\exp[i\alpha y + i\omega t]\,, \tag{6}$$

where m is the radial wavenumber, α is the azimuthal wavenumber and ω is the frequency. Inserting the ansatz (5, 6) into the linearized equations (1, 2), the dispersion equation is obtained

$$\Big(P\,i\,\omega + m^2\pi^2 + \alpha^2\Big) \times$$
$$\times \Big[(i\,\omega + m^2\pi^2 + \alpha^2)\,(m^2\pi^2 + \alpha^2) + Q\,m^2\pi^2 + \eta\,i\,\alpha\Big] = R\,\alpha^2\,. \qquad (7)$$

Solving the real and imaginary parts of dispersion equation (7) yields the analytical expressions for the relations $R = R(\alpha)$ and $\omega = \omega(\alpha)$. In Figs. 3.1-3.3 illustrative plots of dispersion curves $R = R(\alpha)$ and $\omega = \omega(\alpha)$ are shown for various dimensionless parameters P, Q and η. Especially, the function $R = R(\alpha)$ is of interest, where two local minima emerge for sufficiently high values of Q (see Fig. 3.1a) and P (see Fig. 3.2a). The necessary condition for the left local minimum formation is the presence of the magnetic field. Inspecting the dispersion curves behaviour shows that relative position of the local minima and their function value varies with change of the model parameters.

Changing the magnetic field in terms of Q while keeping P, η fixed, for example, reveals the qualitative change in $R = R(\alpha)$ behaviour as shown in Fig. 4a. Note that only the global minimum of $R = R(\alpha)$ corresponds to the convective mode. The symbols A and B will be used to denote the convective modes. Depending on the value of Q (see Fig. 4a) the single modes A or B arise. At certain critical value of Q_c both minima possess the same value of R, corresponding to the double mode $A - B$ convection. This resonance property was studied in (*Revallo and Ševčovič, 2002*) where the asymptotic expression for the critical value of Q_c was derived.

Here we explore the two local minima formation more systematically for the geophysically relevant parameter ranges of P, Q and η. Note that the minimization of $R = R(\alpha)$ for general choice of the model parameters can only be performed numerically using the Newton method. The modes A and B possessing $m = 1$ have been proved to be the most unstable ones. We have computed the boundaries between domains of single minimum and two local minima of $R = R(\alpha)$. The representative results are shown in Figs. 5a,b. For small values of P and Q only single minimum of $R = R(\alpha)$ is possible (below the dashed curves). We have also identified domains of the modes A and B preference (divided by the solid curves). The principal observation is that increasing the rotation rate η enlarges the domain marked by A in Figs. 5a,b. Thus making η larger favours the two local minima formation and the A mode preference. With growing η, stronger magnetic field in terms of large Q is needed to keep the mode B preferred.

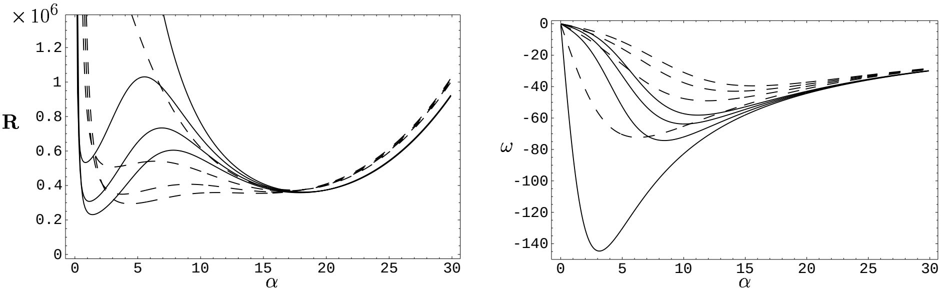

Fig. 3.1 a,b. Dispersion curves $R = R(\alpha)$ and $\omega = \omega(\alpha)$ for the modes $m = 1$ and $m = 2$ (dashed curve) for $P = 10$, $\eta = 10^4$. The curves $R = R(\alpha)$ (top to bottom) and $\omega = \omega(\alpha)$ (bottom to top) correspond to $Q = 0$, 200, 400, 600.

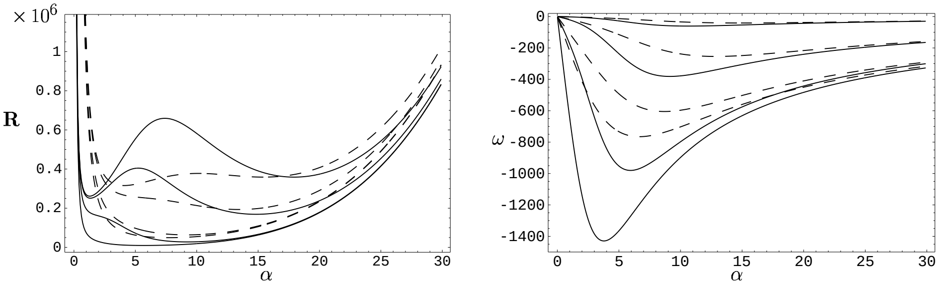

Fig. 3.2 a,b. Dispersion curves $R = R(\alpha)$ and $\omega = \omega(\alpha)$ for the modes $m = 1$ and $m = 2$ (dashed curve) for $\eta = 10^4$, $Q = 500$. The curves $R = R(\alpha)$ (bottom to top) and $\omega = \omega(\alpha)$ (bottom to top) correspond to $P = 0.01$, 0.1, 1, 10.

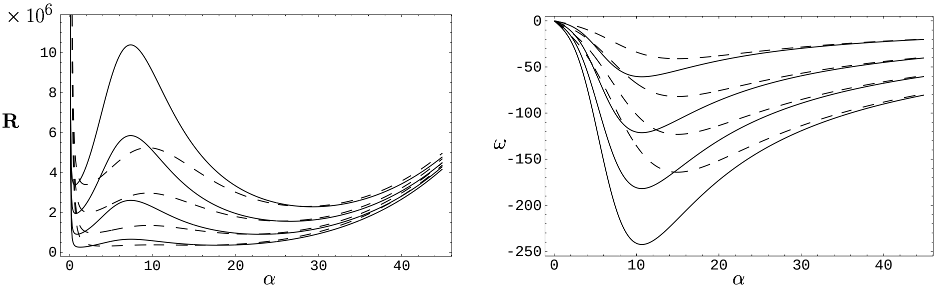

Fig. 3.3 a,b. Dispersion curves $R = R(\alpha)$ and $\omega = \omega(\alpha)$ for the modes $m = 1$ and $m = 2$ (dashed curve) for $P = 10$, $Q = 500$. The curves $R = R(\alpha)$ (bottom to top) and $\omega = \omega(\alpha)$ (top to bottom) correspond to $\eta = 10^4$, 2×10^4, 3×10^4, 4×10^4.

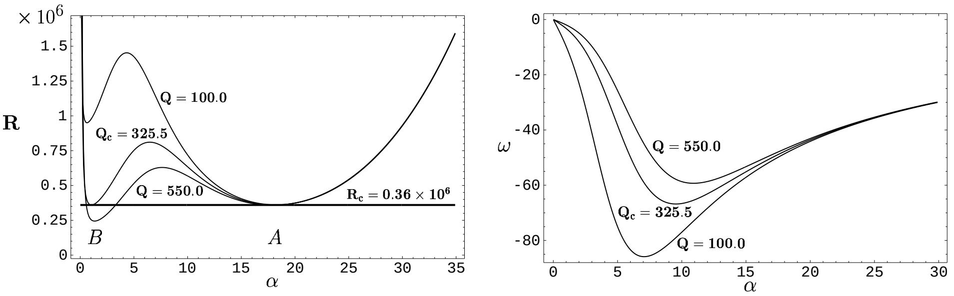

Fig. 4 a,b. Dependencies of $R = R(\alpha)$ and $\omega = \omega(\alpha)$ for $P = 10$, $\eta = 10^4$, for the critical value Q_c and other two values of Q. The two local minima of $R = R(\alpha)$ at $Q_c = 325.5$ correspond to the two most unstable modes A and B *(Revallo and Ševčovič, 2002).* The symbol B identifies the mode which is due to the magnetic field.

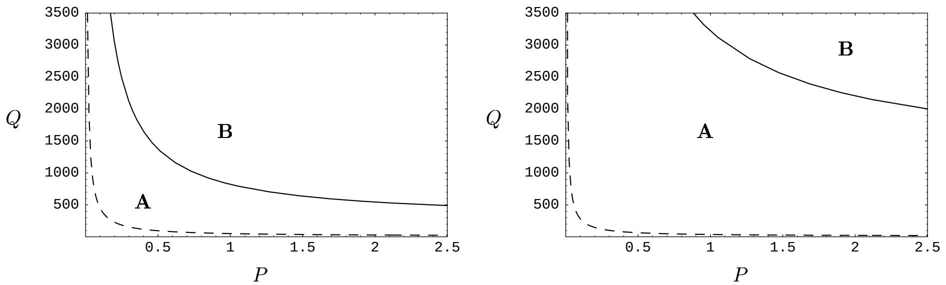

Fig. 5 a,b. Domains of existence of the A and B modes in the $P - Q$ plane for a) $\eta = 10^4$ and b) $\eta = 10^5$. The dashed curves divide the domains of single minimum (below) and two local minima existence. The solid curves correspond to the $A - B$ mode resonance.

4. Conclusions

The two minima dispersion curve phenomenon was explored in dependence on the system parameters, the rotation rate η, the Chandrasekhar number Q and the Prandtl number P. Playing with real and imaginary parts of the dispersion equation was the key analysis tool to obtain the results in Section 3. The plots of dispersion curves in Figs. 3.1-3.3 show that the Rayleigh number R dependence on the azimuthal wavenumber α is rather sensitive to the choice of the system parameters combinations. The radial mode $m = 1$ is sufficient to be considered, as it was proved to be the most unstable one. The number of local minima of $R = R(\alpha)$ and the preference of convective modes depend on the system parameters as shown in Figs. 5a,b.

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