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Geothermal heat flux anomaly due to a 3D prismoid situated in the second layer of a three-layered Earth

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Abstract: We present mathematical modelling of the stationary geothermal field for the three-layered earth which includes a three-dimensional perturbing body below the first layer (over the halfspace substratum). The unperturbed temperature field corresponds to the uniform vertical heat flux. The perturbing body is in the form of 3D prismoid with sloping side faces, while its upper and lower face are rectangles at the planes $z = z_1, z_2$. The theoretical formulae are based on the generalized theory of the double-layer potential and boundary integral equation (BIE). Special attention is paid to the quadrilateral prismoids bounded by planar skew faces. The numerical calculations were performed for the 3D prismoids (blocks), the thermal conductivity of which was greater than that in the ambient second layer, while the upper face of the prismoid may be in contact with the upper layer and the lower face may touch the bottom halfspace. Numerous graphs are shown for the disturbance of the temperature and heat flow distribution on the surface of the Earth or inside all three layers.

Key words: geothermics, heat flux refraction, double layer potential, boundary integral equations, boundary element methods, solid angle of view calculations

1. Introduction

The heat flow from the Earth's interior is of interest in geothermal prospecting based on geothermal models, e.g. *Ljubimova et al. (1983), Chen and Beck (1991), Majcin (1992), Hvoždara and Majcin (2009, 2011), Majcin et al. (2012).* We present the boundary integral solution for the stationary geothermal field in the three layered earth which is perturbed by a 3D body Ω_T of thermal conductivity λ_T situated in the second layer of thermal conductivity λ_2. Below the bottom planar boundary of the second layer we

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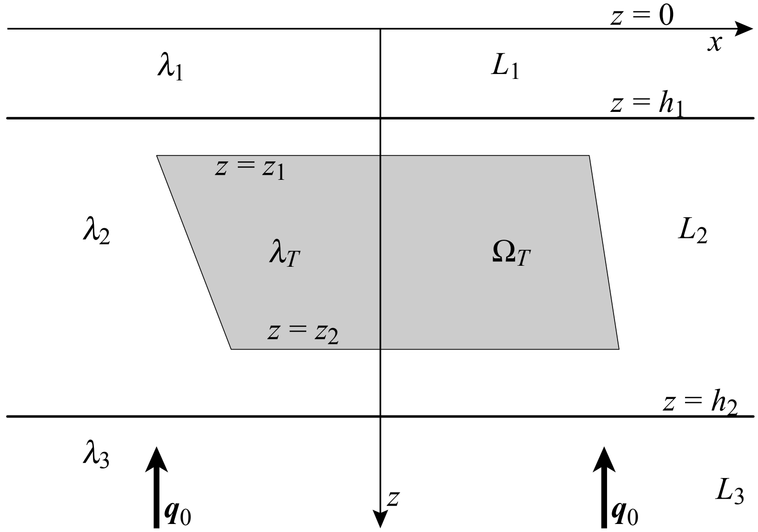


Fig. 1. Model of a 3-D disturbing body buried in the second layer of the three layered earth.

suppose the halfspace substratum $z \geq h_2$ of thermal conductivity λ_3. The cross-section with vertical plane $y = 0$ is depicted in Fig. 1. In the absence of perturbing body in our model we suppose uniform vertical heat flux density q_0 in all three layers. The corresponding temperatures are denoted as $T_k(z),\ k = 1, 2, 3$. These functions obey Laplace equation

$$\nabla^2 T_k(z) = 0. \tag{1}$$

Since considered unperturbed heat flux is independent of x, y we have a simple ordinary differential equation for $T_k(z)$:

$$\mathrm{d}^2 T_k(z)/\,\mathrm{d}\,z^2 = 0. \tag{2}$$

Obviously its solution is linear function of z:

$$T_k(z) = z\,A_k + B_k. \tag{3}$$

Constants A_k, B_k are determined from boundary conditions on boundaries $z = 0, h_1, h_2$:

$$T_1(z)|_{z=0} = 0 \tag{4}$$

$$T_k(z)|_{z=h_k} = T_{k+1}(z)|_{z=h_k}\,, \quad k=1,2 \tag{5}$$

$$[\partial T_k(z)/\partial z]_{z=h_k} = (\lambda_{k+1}/\lambda_k)\,[\partial T_{k+1}(z)/\partial z]_{z=h_k}\,, \quad k=1,2. \tag{6}$$

The boundary condition (4) means that we suppose the temperature of the surface $z=0$ as zero of our temperature scale. For this reason we have $B_1=0$ and then $T_1(z)$ is a linear function of z:

$$T_1(z) = A_1 z. \tag{7}$$

The boundary conditions (5), (6) on the planes $z=h_1,h_2$ give simple linear equations system:

$$\begin{aligned}&A_1 h_1 = A_2 h_1 + B_2, \quad A_1 = (\lambda_2/\lambda_1)A_2,\\ &A_2 h_2 + B_2 = A_3 h_2 + B_3, \quad \lambda_2 A_2 = \lambda_3 A_3 = q_0.\end{aligned} \tag{8}$$

The fourth equation links the heat flux density on the bottom boundary $z=h_2$ to the value q_0 of the heat flux in the bottom halfspace. It means that

$$A_3 = q_0/\lambda_3, \quad A_2 = q_0/\lambda_2 \tag{9}$$

and also

$$A_1 = q_0/\lambda_1. \tag{10}$$

Then we obtain expressions for B_2 and B_3:

$$B_2 = q_0 \frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2} h_1, \tag{11}$$

$$B_3 = B_2 - q_0 \frac{\lambda_2-\lambda_3}{\lambda_2\lambda_3} h_2. \tag{12}$$

Now we have complete algorithm for calculation of the unperturbed temperature field:

$$T_k(P) = T_k(z) = A_k z + B_k, \quad k=1,2,3, \tag{13}$$

where $P \equiv (x, y, z)$ is arbitrary point in the layer L_k. This simple temperature field is changed because of inserted perturbing 3D body Ω_T into layer L_2. Similar problems for two layered earth were in papers *Hvoždara and Valkovič (1999)* and *Hvoždara and Majcin (2011)*. The method of boundary integral equations can be extended also for more complicated three layered earth. We can solve interesting structures, e.g. diapire penetrating from substratum L_3 through layer L_2 to the lower boundary of the layer L_1. Other interesting structure is 3D basin below the layer L_1, buried into L_2.

2. Boundary integral expressions of perturbed temperature field

Using experience from our previous papers *Hvoždara (2008), Hvoždara and Majcin (2011)* mentioned above, we can write temperatures $U_k(P)$ in the k-th layer in the form:

$$U_k(P) = T_k(P) + \frac{1}{4\pi} \int\limits_S f(Q) \frac{\partial}{\partial n_Q} G_{k2}(P, Q) \, \mathrm{d}\, S_Q, \quad P \in L_k, \quad P \notin \Omega_T. \tag{14}$$

In the interior of the perturbing body the temperature is:

$$U_T(P) = \frac{\lambda_2}{\lambda_T} \left[T_2(P) - v_0 + \frac{1}{4\pi} \int\limits_S f(Q) \frac{\partial}{\partial n_Q} G_{22}(P, Q) \, \mathrm{d}\, S_Q \right] + v_0. \tag{15}$$

In formulae (14), (15) the integrals correspond to the effect of the double layer distributed on the surface S of the body Ω_T, their density $f(Q)$ must be calculated by means of BIE

$$f(P) = 2\beta \left[T_2(P) - v_0 \right] + \frac{\beta}{2\pi} \int\limits_S f(Q) \frac{\partial}{\partial n_Q} G_{22}(P, Q) \, \mathrm{d}\, S_Q, \tag{16}$$

where $\beta = (1 - \lambda_T/\lambda_2)/(1 + \lambda_T/\lambda_2)$. The constant v_0 in above formulae is the mean value of the unperturbed temperature $T_2(P)$ on the surface S:

$$v_0 = \frac{1}{S} \int\limits_S T_2(P) \, \mathrm{d}\, S_P. \tag{16a}$$

In formulae (14)–(16) we have normal derivatives ($\partial G/\partial n_Q$) of Green's functions of our potential problem. Similarly, as in *Hvoždara and Majcin (2011)* the Green's function $G_{k2}(P,Q)$ corresponds to the thermal field due to point source of heat situated in the point $Q \in S$ and the temperature is calculated in the point $P \in L_k$. The Green's functions $G_{12}(P,Q)$, $G_{32}(P,Q)$ obey Laplace equations:

$$\nabla^2 G_{12}(P,Q) = 0, \quad \nabla^2 G_{32}(P,Q) = 0 \tag{17}$$

and $G_{22}(P,Q)$ obeys Poisson equation

$$\nabla^2 G_{22}(P,Q) = -4\pi\delta(P,Q), \tag{18}$$

where $\delta(P,Q)$ is Dirac function. These functions must satisfy boundary conditions on $z = 0, h_1, h_2$ similar to the temperature field:

$$G_{12}(P,Q)\mid_{z=0} = 0, \tag{19}$$

$$G_{12}(P,Q)\mid_{z=h_1} = G_{22}(P,Q)\mid_{z=h_1}, \tag{20}$$

$$\lambda_1 \left[\partial G_{12}(P,Q)/\partial z\right]_{z=h_1} = \left[\lambda_2 \partial G_{22}(P,Q)/\partial z\right]_{z=h_1}. \tag{21}$$

$$G_{22}(P,Q)\mid_{z=h_2} = G_{32}(P,Q)\mid_{z=h_2}, \tag{22}$$

$$\lambda_2 \left[\partial G_{22}(P,Q)/\partial z\right]_{z=h_2} = \lambda_3 \left[\partial G_{32}(P,Q)/\partial z\right]_{z=h_2}. \tag{23}$$

All three Green's function must decrease to zero for $\overline{PQ} \to \infty$:

$$\lim_{\overline{PQ}\to\infty} G_{k2}(P,Q) = 0. \tag{24}$$

The Cartesian coordinates for points P, Q are as follows:

$$P \equiv (x, y, z), \quad Q \equiv (x', y', z')$$

The Poisson equation (18) tells us that principal term in $G_{22}(P,Q)$ is a well known point source potential

$$R^{-1} = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-1/2}, \tag{25}$$

because

$$\nabla^2(R^{-1}) = -4\pi\delta(P, Q). \tag{26}$$

Then we have for $G_{22}(P, Q)$ the expression:

$$G_{22}(P, Q) = R^{-1} + \tilde{G}_{22}(P, Q), \tag{27}$$

where $\tilde{G}_{22}(P, Q)$ is harmonic function

$$\nabla^2 \tilde{G}_{22}(P, Q) = 0. \tag{28}$$

Now we introduce an auxiliary cylindrical system (r, z) with vertical polar axis z running through the point Q, and we put

$$r = \left[(x - x')^2 + (y - y')^2\right]^{1/2}. \tag{29}$$

Then the solutions G_{12}, $\tilde{G}_{22}$, G_{32} satisfy the well known Laplace equation in cylindrical coordinates (r, z):

$$\nabla^2 G(r, z) = 0, \quad \text{i.e.}$$
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{\partial^2 G}{\partial z^2} = 0. \tag{30}$$

Its general solution for the z-layered earth is known in geophysical potential problems:

$$G(r, z) = \int_0^\infty \left[Ce^{tz} + De^{-tz}\right] J_0(tr)\,\mathrm{d}\,t, \tag{31}$$

where $J_0(tr)$ is Bessel function of the first kind, zero index. In this form we know also the expression for R^{-1} (by means of Weber-Lipschitz integral):

$$R^{-1} = \left[r^2 + (z - z')^2\right]^{-1/2} = \int_0^\infty e^{-t|z-z'|} J_0(tr)\,\mathrm{d}\,t. \tag{32}$$

It is necessary to put $|z - z'| = z - z'$ for $z > z'$ and $|z - z'| = z' - z$ for $z < z'$ in the layer L_2. In view of the boundary condition (19) we suppose for the Green function $G_{12}(r, z)$ the expression:

$$G_{12}(r, z) = \int_0^\infty C_1 \operatorname{sh}(tz) J_0(tr)\,\mathrm{d}\,t, \tag{33}$$

because $\mathrm{sh}(tz) = 0$ at the plane $z = 0$. For $G_{22}(r, z)$ we have the expression:

$$G_{22}(r,z) = \int_0^\infty \left[e^{-t|z-z'|} + C_2 e^{tz} + D_2 e^{-tz} \right] J_0(tr) \, \mathrm{d}\, t. \tag{34}$$

Because in the substratum L_3 the z-ordinate can grow to infinity, the function e^{tz} cannot occur in $G_{32}(r, z)$ and we have

$$G_{32}(r,z) = \int_0^\infty C_3 e^{-tz} J_0(tr) \, \mathrm{d}\, t. \tag{35}$$

The coefficients C_1, C_2, D_2, C_3 will be functions of the integral variable t and we then determine the boundary conditions (20)–(23) which must hold for all distances r. Then we obtain four linear equations:

$$\begin{aligned}
&C_1 \, \mathrm{sh}(th_1) = e^{-t(z'-h_1)} + C_2 e^{th_1} + D_2 e^{-th_1}, \\
&(\lambda_1/\lambda_2) C_1 \, \mathrm{ch}(th_1) = e^{-t(z'-h_1)} + C_2 e^{th_1} - D_2 e^{-th_1}, \\
&C_3 e^{-th_2} = e^{-t(h_2-z')} + C_2 e^{th_2} + D_2 e^{-th_2}, \\
&-(\lambda_3/\lambda_2) C_3 e^{-th_2} = -e^{-t(h_2-z')} + C_2 e^{th_2} - D_2 e^{-th_2}.
\end{aligned} \tag{36a-d}$$

The elimination method gives at first the coefficients D_2, C_2:

$$\begin{aligned}
D_2 = &\left[k_{12} e^{-t(z'-2h_1)} - e^{-tz'} + k_{12} k_{32} e^{-t(2h_2-2h_1-z')} - \right. \\
&\left. - \, k_{32} e^{-t(2h_2-z')} \right] [W(t)]^{-1},
\end{aligned} \tag{37}$$

where $k_{12} = (1 - \lambda_1/\lambda_2)/(1 + \lambda_1/\lambda_2)$, $k_{32} = (1 - \lambda_3/\lambda_2)/(1 + \lambda_3/\lambda_2)$

and the auxiliary function:

$$W(t) = 1 - k_{12} e^{-2th_1} + k_{32} e^{-2th_2} - k_{12} k_{32} e^{-2t(h_2-h_1)}, \tag{38}$$

$$\begin{aligned}
C_2 = k_{32} &\left[e^{-t(2h_2-z')} - k_{12} e^{-t(2h_2+2h_1-z')} + \right. \\
&\left. + \, k_{12} e^{-t(2h_2-2h_1+z')} - e^{-t(2h_2+z')} \right] [W(t)]^{-1},
\end{aligned} \tag{39}$$

$$C_3 = (1 + k_{32}) \left[e^{tz'} - k_{12} e^{-t(2h_1-z')} + k_{12} e^{t(2h_1-z')} - e^{-tz'} \right] [W(t)]^{-1}, \tag{40}$$

$$C_1 = 2(1 + k_{12}) \left[e^{-tz'} + k_{32} e^{-t(2h_2 - z')} \right] [W(t)]^{-1} . \tag{41}$$

We can see that all coefficients have the same denominator, i.e. multiplicative function $[W(t)]^{-1}$. This function can be expanded into infinite geometrical series if the depths h_1, h_2 are integer multiple of some common depth scale D, i.e. we suppose:

$$h_1 = i_1 D, \quad h_2 = i_2 D, \tag{42}$$

while $i_1 \geq 1$, $i_2 > i_1$ are integers. It is clear that coefficients k_{12} and k_{32} are in absolute values less than 1, and so we can express $[W(t)]^{-1}$ given by (38) in modified form:

$$\begin{aligned}[W(t)]^{-1} &= \left[1 - f_1 e^{-t2i_1 D} - f_2 e^{-t2i_2 D} - f_3 e^{-t2(i_2 - i_1)D} \right]^{-1} = \\ &= \left[1 - f_1 g^{i_1} - f_2 g^{i_2} - f_3 g^{i_3} \right]^{-1} ,\end{aligned} \tag{43}$$

where we introduced notations

$$f_1 = k_{12},\ f_2 = -k_{32},\ f_3 = k_{12} k_{32},\ i_3 = i_2 - i_1 \text{ and } g = e^{-2Dt}. \tag{44}$$

By using knowledge from DC geoelectric problems e.g. *Bhattacharya and Patra (1968)* we find expansion:

$$\left[1 - f_1 g^{i_1} - f_2 g^{i_2} - f_3 g^{i_2 - i_1} \right]^{-1} = \sum_{n=0}^{\infty} s_n g^n , \tag{45}$$

where $g < 1$ and also $|f_1| < 1$, $|f_2| < 1$, $|f_3| < 1$. The relation (45) will be satisfied if there holds:

$$1 = \left[1 - f_1 g^{i_1} - f_2 g^{i_2} - f_3 g^{i_3} \right] \sum_{n-0}^{\infty} s_n g^n . \tag{46}$$

Now we multiply individual terms in square brackets with the infinite sum and obtain

$$1 = \sum_{n=0}^{\infty} s_n g^n - f_1 \sum_{n=0}^{\infty} s_n g^{n+i_1} - f_2 \sum_{n=0}^{\infty} s_n g^{n+i_2} - f_3 \sum_{n=0}^{\infty} s_n g^{n+i_2-i_1} . \tag{47}$$

If we perform suitable shift of summation coefficients in the second, third and fourth series and compare the resulting series with value 1 on the l.h.s. in (47) we obtain recurrence relations for coefficients s_n:

$$\begin{aligned} & s_0 = 1, \\ & s_n = f_1 s_{n-i_1} + f_3 s_{n-i_2+n_1} \text{ for } n = 1, 2, \dots n_1, \\ & \qquad\qquad\qquad\qquad \text{where } \; n_1 = \max[i_1, i_2 - i_1] \\ & s_n = f_1 s_{n-i_1} + f_2 s_{n-i_2} + f_3 s_{n-i_2+i_1} \text{ for } n > n_1. \end{aligned} \tag{48}$$

Note that we take $s_k \equiv 0$ for $k < 0$. Using the coefficients s_n we can use expansion:

$$[W(t)]^{-1} = \sum_{n=0}^{\infty} s_n e^{-2tnD} \tag{49}$$

in all integrals with C_1, C_2, D_2, C_3 in Green's functions. Numerical calculations showed that $|s_n| < 1$, for $n \geq 1$, so the series (49) is rapidly convergent. Now we can present explicit expressions for Green's functions $G_{12}(P,Q)$, $G_{22}(P,Q)$, $G_{32}(P,Q)$. In all integrals we can use the known Weber-Lipschitz integral:

$$\int_0^{\infty} e^{-t\xi} J_0(tr)\, \mathrm{d}\, t = \left[r^2 + \xi^2\right]^{-1/2}, \text{ provided } \xi > 0. \tag{50}$$

By using expression (41) for C_1 and expansion (49) we obtain from (33) for the layer $L_1 : \; z \in \langle 0, h_1 \rangle, z' > h_1$:

$$\begin{aligned} G_{12}(P,Q) = (1 + k_{12}) \sum_{n=0}^{\infty} s_n & \left\{ \left[r^2 + (2nD + z' - z)^2\right]^{-1/2} - \right. \\ & \left. - \left[r^2 + (2nD + z' + z)^2\right]^{-1/2} \right\} + (1 + k_{12}) k_{32} \sum_{n=0}^{\infty} s_n \cdot \\ & \cdot \left\{ \left[r^2 + (2nD + 2h_2 - z' - z)^2\right]^{-1/2} - \right. \\ & \left. - \left[r^2 + (2nD + 2h_2 - z' + z)^2\right]^{-1/2} \right\}. \end{aligned} \tag{51}$$

For the layer L_2 : $z \in \langle h_1, h_2\rangle$, $z' \in \langle h_1, h_2\rangle$ we have a more complicated Green's function $G_{22}(P,Q)$ using (34), (37), (39) and (49):

$$G_{22}(P,Q) = R^{-1} + k_{32}\sum_{n=0}^{\infty} s_n \left\{ \left[r^2 + (2nD + 2h_2 - z' - z)^2\right]^{-1/2} - \right.$$
$$- k_{12}\left[r^2 + (2nD + 2h_2 + 2h_1 - z' - z)^2\right]^{-1/2} -$$
$$- \left[r^2 + (2nD + 2h_2 + z' - z)^2\right]^{-1/2} +$$
$$\left. + k_{12}\left[r^2 + (2nD + 2h_2 - 2h_1 + z' - z)^2\right]^{-1/2} \right\} +$$
$$+ \sum_{n=0}^{\infty} s_n \left\{ k_{12}\left[r^2 + (2nD - 2h_1 + z' + z)^2\right]^{-1/2} - \right.$$
$$- \left[r^2 + (2nD + z' + z)^2\right]^{-1/2} +$$
$$+ k_{12}k_{32}\left[r^2 + (2nD + 2h_2 - 2h_1 - z' + z)^2\right]^{-1/2} -$$
$$\left. - k_{32}\left[r^2 + (2nD + 2h_2 - z' + z)^2\right]^{-1/2} \right\}. \tag{52}$$

Similarly for the substratum halfspace $L_3 (z > h_2)$ we have from (35) using (40) and (49):

$$G_{32}(P,Q) = (1 + k_{32})\sum_{n=0}^{\infty} s_n \left\{ \left[r^2 + (2nD + z - z')^2\right]^{-1/2} - \right.$$
$$- k_{12}\left[r^2 + (2nD + 2h_1 - z' + z)^2\right]^{-1/2} +$$
$$+ k_{12}\left[r^2 + (2nD - 2h_1 + z' + z)^2\right]^{-1/2} -$$
$$\left. - \left[r^2 + (2nD + z' + z)^2\right]^{-1/2} \right\}. \tag{53}$$

We must remember that $r^2 = (x - x')^2 + (y - y')^2$ is a square of horizontal distance of points PQ. One can easily check that in the summation part there are no singular terms even for $n = 0$ if $h_1 < z' < h_2$. The contact cases, i.e. if the body Ω_T touches with some upper planar part the plane $z = h_1$ or with bottom planar part the boundary $z = h_2$ will be considered similarly as in *Hvoždara and Valkovič (1999), Hvoždara and Majcin (2011).*

The analysis of Green's function $G_{22}(P,Q)$ given by the complex formula (52) shows that if the perturbing body touches the upper plane $z = h_1$we have two singular terms, namely:

$$R^{-1} + k_{12}\left[r^2 + (2h_1 - z - z')^2\right]^{-1/2}, \tag{54}$$

where we used term for $n = 0$ of the second infinite series, while $s_0 = 1$. By using limit approach as explained in *Hvoždara and Majcin (2011)* we can show that in the BIE (16) for contact points $P \in S_{h_1}$ we must instead of β use the coefficients

$$\beta_1 = \beta/(1 - \beta k_{12}). \tag{55}$$

Analogously, for the contact case with the lower planar boundary $z = h_2$ we have also two singular terms in $G_{22}(P,Q)$, namely

$$R^{-1} + k_{32}\left[r^2 + (2h_2 - z - z')^2\right]^{-1/2}, \tag{56}$$

where we used the term for $n = 0$ of the first infinite series, while $s_0 = 1$. By using limit approach as explained in *Hvoždara and Valkovič (1999)* we can show that in the BIE (16) for contact points $P \in S_{h_2}$ we must instead of β use coefficient

$$\beta_2 = \beta/(1 - \beta k_{32}). \tag{57}$$

In this manner we can write instead of BIE (16) a more general form which includes also the possibility of contact cases:

$$f(P) = 2\gamma\left[T_2(P) - v_0\right] + \frac{\gamma}{2\pi}\int\!\!\!\!\!\!\!\!/\!\!/_{S} f(Q)\frac{\partial}{\partial n_Q}G_{22}(P,Q)\,\mathrm{d}\,S_Q, \tag{58}$$

where

$$\gamma = \begin{cases} \beta, & \text{if } P \notin S_{h_1}, P \notin S_{h_2} \\ \beta/(1 - \beta k_{12}) & \text{if } P \in S_{h_1} \\ \beta/(1 - \beta k_{32}) & \text{if } P \in S_{h_2} \end{cases}. \tag{59}$$

The double slash in the integral sign of BIE (58) denotes that for $P \in S_{h_1}$ there are omitted contributions from function (54) and similarly for $P \in S_{h_2}$ there are omitted contribution from function (56), while for the rest of the surface S there are omitted only contributions due to R^{-1}. Concerning the formula (59) we note that there appears normal derivative of the Green's function $G_{22}(P,Q)$. If the unit normal vector at the point $Q \equiv (x', y', z')$ is

$$\boldsymbol{n}_Q \equiv (n'_x, n'_y, n'_z), \tag{60}$$

then the normal derivative is

$$\frac{\partial G_{22}(P,Q)}{\partial n_Q} = n'_x \frac{\partial G_{22}}{\partial x'} + n'_y \frac{\partial G_{22}}{\partial y'} + n'_z \frac{\partial G_{22}}{\partial z'}. \tag{61}$$

3. Numerical calculations and discussion

The numerical calculations were performed for the prismoid with the upper face of the rectangular form at the depth $z_1 \in \langle h_1, h_2 \rangle$. The sides of the rectangle were considered to be parallel to the x, y axes, while: $x \in \langle x_{\ell_1}, x_{r_1} \rangle$, $y \in \langle y_{\ell_1}, y_{r_1} \rangle$. The bottom face is also rectangle at the depth z_2, while $z_1 < z_2 \leq h_2$. The sides of the bottom rectangle we consider also parallel to the x, y axes, while: $x \in \langle x_{\ell_2}, x_{r_2} \rangle$, $y \in \langle y_{\ell_2}, y_{r_2} \rangle$. It is clear that $x = \text{const}$, $y = \text{const}$ sides of the lower and upper rectangle are parallel to each other. Then the side faces of the prism are trapezoids form with upper segment at the depth $z = z_1$, bottom segment at the depth $z = z_2$ and the sides of the trapezoid are skew segments connecting the upper and bottom segment. In such definition of the prismoid faces we have on each face constant direction of the normal vector $\boldsymbol{n}_Q$ and possibility of simple manner of subdivision onto a set of subareas. According to experience from our previous paper *Hvoždara and Majcin (2011)*, we adopted subdivision of each face onto $a \times a$ subareas ΔS_j (here we applied $a = 12$ or $a = 16$) and obviously we have total number $M = 6 \times a \times a$ of trapezoidal subareas ΔS_j. The BIE (58) we solve by the collocation method, i.e. we suppose on each subarea ΔS_j the unknown function $f(Q)$ to be constant $f(Q_j)$ where Q_j is the central point of ΔS_j. In this manner we introduce the piecewise constant approximation of the unknown double layer density $f(Q)$ on the surface S. Putting the number M sufficiently large we can express the BIE (58) in its discretized form:

$$f(P_m) = 2\gamma\,[T_2(P) - v_0] + \sum_{j=1}^{M} f(Q_j) w_2(P_m, Q_j), \quad m = 1, 2, \ldots, M. \tag{62}$$

Here $\gamma = \beta$ if the body does not touch at the point P_m the plane boundaries $z = h_1, h_2$ and attains slightly changed values for the contact cases as show by formula (58a). We introduced the weighting coefficients $w_2(P_m, Q_j)$ in the following manner:

$$w_2(P_m, Q_j) = \frac{\gamma}{2\pi} \underset{\Delta S_j}{\fint} \frac{\partial}{\partial n_Q} G_{22}(P_m, Q) \, \mathrm{d}\, S_Q. \tag{63}$$

The integration in the principal value sense means that there are excluded contributions from R^{-1} if P_m is normal point or contributions from function (54) if $P_m \in S_{h_1}$ or from function (56) if $P_m \in S_{h_2}$. The formula (61) can be written as a system of M linear equations for unknowns $f(Q_j)$ in the next manner:

$$\sum_{j=1}^{M} \left[\delta_{mj} - w_2(P_m, Q_j)\right] f(Q_j) = 2\gamma \left[T_2(P_m) - v_0\right], \tag{64}$$

where δ_{mj} is the Kronecker's symbol. Then the system of equations can be solved using methods of linear algebra and standard computer subroutines. Once the system (64) is solved, we can calculate the temperature field and also the vertical component of the heat flux or their anomalies ΔT, Δq_z. For this purpose we employ summation approximation of formulae (14), (15). Then for the layer L_k we have for the temperature $U_k(P)$:

$$U_k(P) = T_k(P) + \sum_{j=1}^{M} f(Q_j) X_k(P, Q_j), \tag{65}$$

where

$$X_k(P, Q_j) = \frac{1}{4\pi} \int\limits_{\Delta S_j} \frac{\partial G_{k2}(P, Q)}{\partial n_Q} \, \mathrm{d}\, S_Q \tag{66}$$

is the weighting factor of influence the dipole seated at the subarea ΔS_j. Similar formula can be created also for calculation of the temperature $U_T(P)$ in the interior of perturbing body:

$$U_T(P) = (\lambda_2/\lambda_T) \left[T_2(P) - v_0 + \sum_{j=1}^{M} f(Q_j) X_2(P, Q_j)\right] + v_0. \tag{67}$$

Let us note that crucial role in numerical calculation of the coefficients $w_2(P_m, Q_j)$ for solution of the BIE (61) and also coefficients $X_k(P, Q_j)$ given by (65). The principal term in all Green's functions is

$$R^{-1} = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-1/2}.$$

The integral of this term over trapezoidal subarea ΔS_j was calculated by means of our original subroutine SLAGUP4, which is adopted by using formulae by *Guptasarma and Singh (1999).* Our algorithm is sufficiently explained in the paper *Hvoždara and Majcin (2011).* This subroutine is calculating the solid angle of view based on the vector and scalar products of four vectors connecting the point $P \equiv (x, y, z)$ with four corners of trapezoidal planar subarea ΔS_j with outer normal $\boldsymbol{n}_Q$. This subroutine can be adopted also for precise calculation of contributions of the terms like

$$R_\zeta^{-1} = \left[(x-x')^2 + (y-y')^2 + (\zeta-z')^2\right]^{-1/2}, \tag{68}$$

where $\zeta = 2h_{1,2} - z$ because at the contact cases this term attains also high values. In such a manner we can calculate the temperature field $U_k(x, y_c, z)$ for the constant y_c plane. We obtain maps of isotherms in the (x, y) plane. If we use suitable small depth internal Δz, then we can also calculate the vertical heat flux density q_z by the difference of temperatures at neighbouring levels z:

$$q_z(x, y, z_j) = \lambda_k \left[U_k(x, y, z_j + \Delta z) - U_k(x, y, z_j)\right] / \Delta z, \tag{69}$$

where $\Delta z \doteq h_1/20$.

We investigated a number of models, but here we present graphical results for two models only. The first model approximates prismoidal depression of the first layer into second one. In this model we put thickness of the layer L_1 as a unit depth scale, i.e. $h_1 = D = 1$ and $h_2 = 2h_1$, so we have $i_1 = 1$, $i_2 = 2$ in the expansion in the formula (45). The top face of the prism is rectangle in contact with the layer L_1, i.e. $z_1 = h_1$ and the bottom face is also smaller rectangle at the depth $z_2 = 1.8h_1$, so the prism is quite thick, since $z_2 - z_1 = 0.8h_1$. The cross-section of the prismoid by the plane $y = 0$ is trapezoid depicted in the top of Fig. 2a, together with isolines of relative temperature (isotherms) in the x, z-variables, while $y = 0$. The geometrical parameters are presented in the first two rows in the table below.

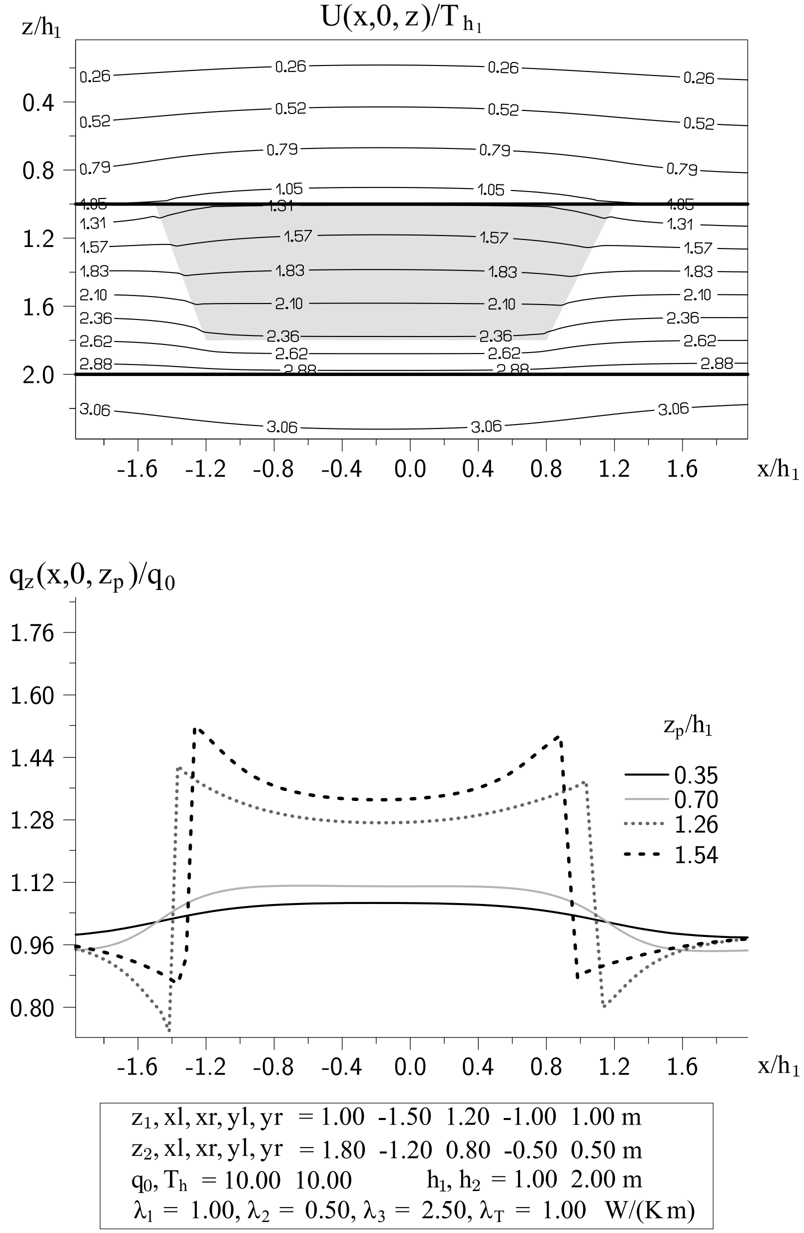
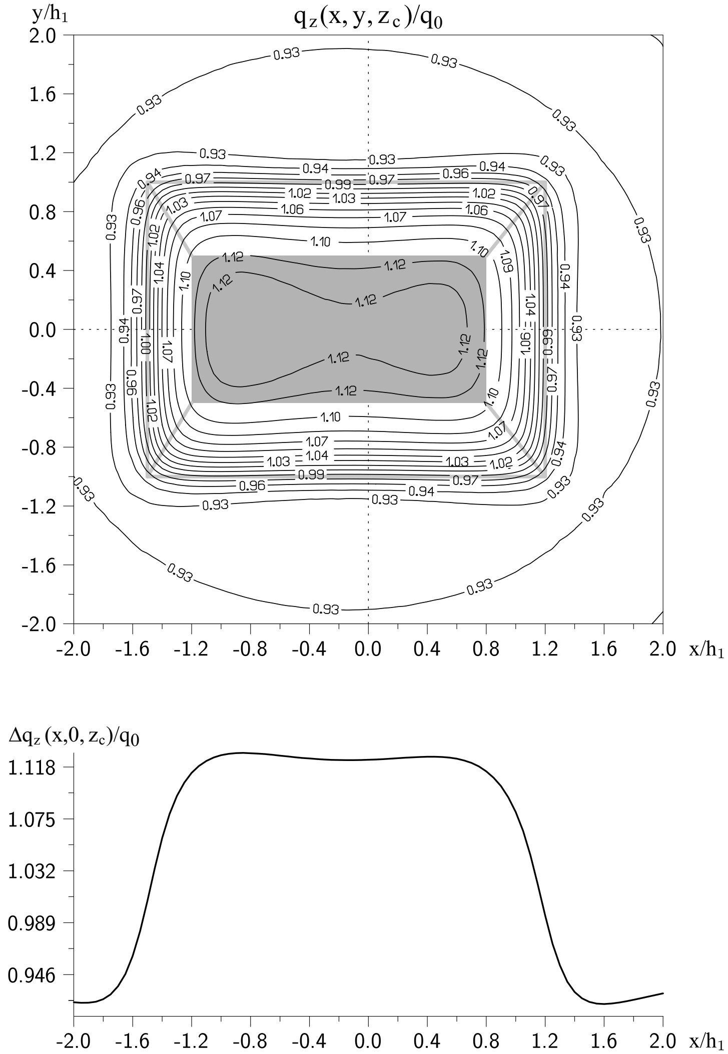


Fig. 2a. Isolines of the relative temperature (top) and profile curves of the relative vertical heat flux (bottom) in the plane $y = 0$ for the prismoidal depression of the first layer into second one while $\lambda_T = \lambda_1$, $\lambda_T/\lambda_2 = 2$, $\lambda_3/\lambda_1 = 2.5$ and $h_2 - h_1 = h_1 = 1$.



z_1, xl, xr, yl, yr = 1.00 -1.50 1.20 -1.00 1.00 m
z_2, xl, xr, yl, yr = 1.80 -1.20 0.80 -0.50 0.50 m
q_0, T_h = 10.00 10.00 h_1, h_2, z_c = 1.00 2.00 0.80 m
λ_1 = 1.00, λ_2 = 0.50, λ_3 = 2.50, λ_T = 1.00 W/(K m)

Fig. 2b. Isolines of the relative vertical heat flux at the plane $z_c/h_1 = 0.8$ (top) and profile curve at $y = 0$ in the first layer for the the same parameters as in Fig. 2a. There is also depicted projection of the prismoid with sloped faces, the gray rectangle is projection of the bottom face.

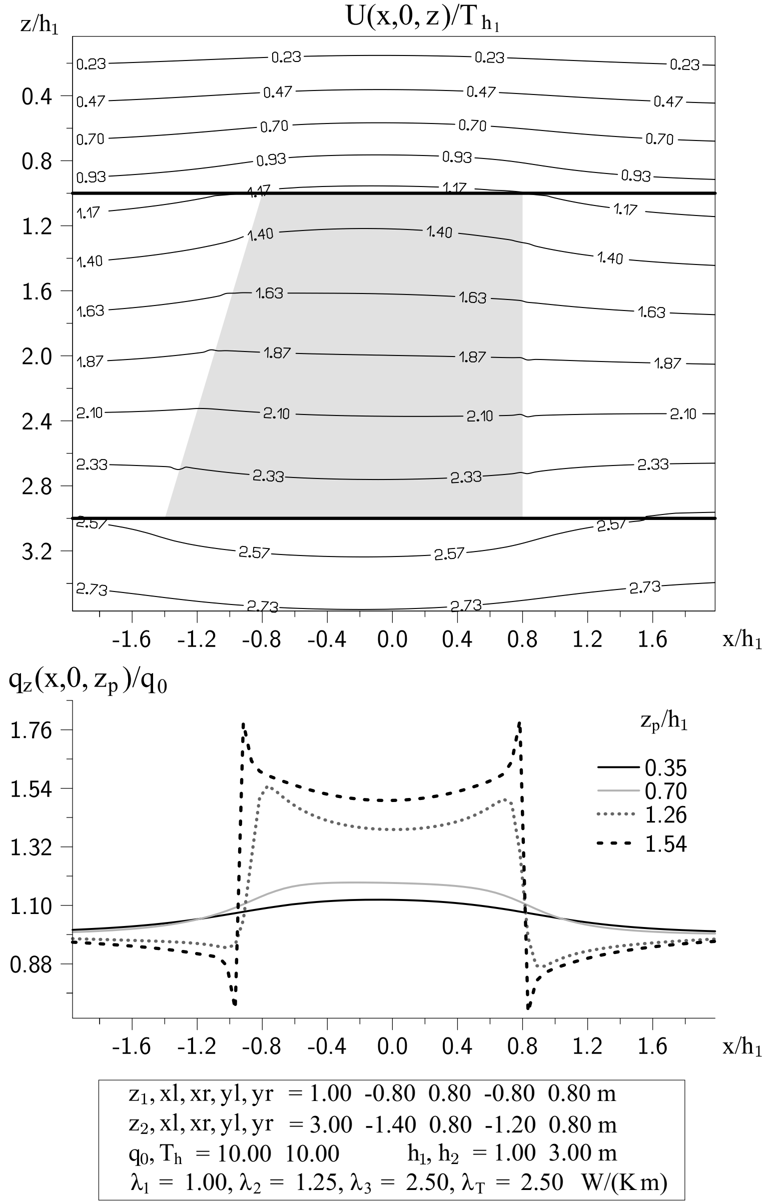
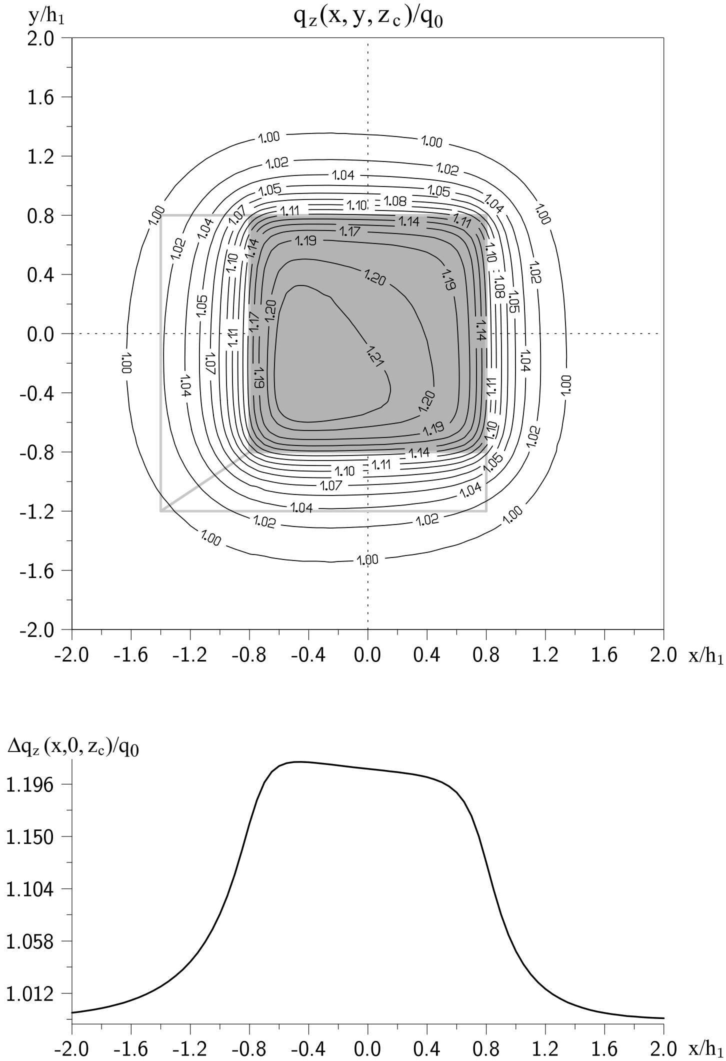


Fig. 3a. Isolines of the relative temperature (top) and profile curves of the relative vertical heat flux (bottom) in the plane $y = 0$ for the prismoidal diapire from the substratum till the bottom boundary $z = h_1$ of the first layer. According to the data given in the table there is $\lambda_T/\lambda_2 = \lambda_3/\lambda_2 = 2$, $\lambda_2/\lambda_1 = 1.25$, $h_2 - h_1 = 2h_1$.



z_1, xl, xr, yl, yr = 1.00 -0.80 0.80 -0.80 0.80 m
z_2, xl, xr, yl, yr = 3.00 -1.40 0.80 -1.20 0.80 m
q_0, T_h = 10.00 10.00 h_1, h_2, z_c = 1.00 3.00 0.80 m
λ_1 = 1.00, λ_2 = 1.25, λ_3 = 2.50, λ_T = 2.50 W/(K m)

Fig. 3b. Isolines of the relative vertical heat flux at the plane $z_c/h_1 = 0.8$ (top) and profile curve at $y = 0$ in the first layer for the the same parameters of the diapire as in Fig. 3a. There is also depicted projection of the prismoid with sloped faces, the gray square is projection of the upper face of the diapire.

The four profile curves present the vertical heat flux $q_z(x, 0, z_p)$ normed by q_0 at the depths $z_p/h_1 = 0.35,\ 0.70,\ 1.26,\ 1.54$. In the table we can also see that $\lambda_T/\lambda_1 = 1$, while $\lambda_2/\lambda_1 = 0.5$ and $\lambda_3/\lambda_1 = 2.5$. We can see that heat flux anomalies are about 10% of q_0 if $z_p/h_1 < 1$, but two profiles for $z_p/h_1 > 1$ show anomalies $q_z/q_0 \approx 1.4$ if the calculation points lie in the interior of the prism, since it is surrounded by the less conductive material of L_2. One can also see the effect of draining of the heat flux from L_2 into more conductive prismoid if the calculation point approaches the prismoid from the outside. In Fig. 2b we can see isolines of the normed vertical heat flux distribution above the prismoid at the depth $z_c = 0.8h_1$. The gray rectangle depictes the bottom face of the prismoidal depression. We can see that the values of $q_z(x, y, z_c)/q_0$ attain cca 1.12 above the central region of the prismoid.

The results for the second model are presented in Figs. 3a,b. The prismoidal perturbing body approximates the diapire connecting the substratum halfspace L_3 with the bottom boundary of the layer L_1. In this model we put $h_2 = 3h_1$, so $i_1 = 1$ and $i_2 = 3$ in the expansion (45). The thermal conductivities we put $\lambda_1 = 1$, $\lambda_2 = 1.25$, $\lambda_3 = 2.5$, $\lambda_T = 2.5\,\mathrm{W/(K\,m)}$. The geometrical parameters are given in the first two rows in the table of Fig. 3a. The cross-section of the prismoid by the plane $y = 0$ is trapezoid depicted in the top of Fig. 3a, together with isolines of temperature in the x, z-variables, while $y = 0$. The temperature values are normed by the normal temperature T_1 at the depth h_1. From the profile curves of q_z/q_0 for depths $z_p/h_1 = 0.35$, 0.70, 1.26, 1.54 we can see that the heat flux anomaly is greater than in the previous model, clearly due to both by greater volume of good conducting prismoid and by its shape with prevailing vertical dimension and smaller upper face. In Fig. 3b we can see isolines of the vertical heat flux above the diapire at the depth $z_c = 0.8h_1$. The gray square depictes the upper face of the diapire at the boundary $z = h_1$. The asymmetry of isolines is clearly due to various slope of the side faces. By using results of presented models we can conclude that the most useful region for exploating the heat flux anomalies is region in L_1 above the anomalous good conducting body or inside it. It is clear that the localization of the anomalous body Ω_T and its dimensions must be determined by application of geophysical exploration methods: gravimetric, magnetometric, geoelectric as well as by reflex seismic profiling.

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