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Distributed Fusion Estimation for the Measurements with Bounded Disturbances

Qiang Shen, Can Li, Jieyu Liu, Xinsan Li, Lixin Wang

Xi'an Research Institute of High-Tech, Xi'an 710025, China, sheng110@163.com

Abstract: The information fusion problem is studied for multi-sensor systems in the presence of bounded disturbances. In this paper, a distributed fusion estimation algorithm is proposed based on the set-membership theory, which obtains the overall estimates based on multi-ellipsoids intersection. A parameter adaptive adjustment scheme is derived to guarantee the performance of the algorithm. The feedback mechanism is also introduced to enhance the estimation procedure. Through theoretical analysis and simulation, the performance of the proposed algorithm is analyzed, and some interesting properties of the proposed algorithm are proved. Results show that the proposed algorithm improves the point estimation accuracy. Compared with the algorithm without feedback, the one with feedback has better local estimation. Meanwhile, the effectiveness of the proposed algorithm in improving state estimation accuracy has been proved by the simulation results.

Keywords: distributed fusion, bounded disturbances, set-membership estimation, ellipsoidal state bounding.

1. INTRODUCTION

In the measurement field, the functional requirements of large and complex systems are rapidly increasing in recent years. When using a single sensor, obvious defects exist in measurement accuracy, stability, and reliability. Thus, multi-sensor systems and related information fusion technologies have attracted more and more attention, which are widely used in measurement applications [1]-[4]. The key issue of multi-sensor estimation fusion is how to fuse the measurement data from multiple sensors to provide more useful and accurate state estimation results [5], [6].

Most existing information fusion algorithms are based on probability and require accurate noise statistics. This idealized assumption is difficult to meet in some applications, which may lead to a decrease in the performance of state estimation. Exactly, it is easier to obtain the bounds of noise with unknown statistics. Therefore, set-membership algorithms in which the noises are only assumed to be bounded provide an interesting alternative and have recently attracted more and more attention [7]-[8]. Set-membership theory has been widely used, including automatic control [9]-[10], faulty detection [11], simultaneous localization and mapping (SLAM) [12], etc. For multi-sensor fusion with bounded disturbances, Becis-Aubry proposed a hierarchical set-membership estimation algorithm equipped with a local processor [13]. Then it is extended for a nonlinear system with potentially failing measurements [14]. In [15], the set-membership information fusion problem for multisensory nonlinear dynamic systems was converted into a semidefinite programming problem, which was solved by using decoupling technique. It is worth mentioning that a combined information filtering in multisensory systems is presented in [16] to better utilize the potentials of both stochastic and set-membership concepts. Now the set-membership fusion approaches have been successfully applied in sensor networks [17], [18] and positioning [19], [20]. However, compared with the probability-based fusion filtering, the set-membership-based information fusion has not received enough attention.

Information fusion has two typical architectures, centralized fusion [21] and distributed fusion [13]. Due to the rapid advances in sensor and communication technologies, the demand for distributed implementations of estimation algorithms is steadily increasing, which is the focus of this paper. When the ellipsoids estimated by the local processors transmit simultaneously to the fusion center, the first thread is to obtain an out bounding ellipsoid enclosing the intersection of these ellipsoids. Then a distributed fusion algorithm based on multi-ellipsoids intersection is proposed. And a novel selection method of the weighting parameters is presented. We compare the proposed algorithm in this paper and the algorithm proposed by Becis-Aubry [13]. In addition, the algorithms with feedback are also studied.

The paper is organized as follows. Section 2 states the problem formulation for the distributed fusion with bounded setting. The distributed set-membership fusion algorithm is

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derived in Section 3, and the parameter adjustment methods are given. In Section 4, some properties of the proposed algorithm are proved. A numerical example is used to prove the effectiveness and properties of the algorithms in Section 5. Section 6 summarizes this article.

2. PROBLEM STATEMENT

Definition 1.

A set bounded by an ellipsoid can be described by

$$\mathcal{E}(a,M)=\{x\in\mathbb{R}^n:(x-a)^T M^{-1}(x-a)\le 1\} \tag{1}$$

where $a\in\mathbb{R}^n$ is the center, and $M\in\mathbb{R}^{n\times n}$ is a positive-definite matrix, which specifies the size and orientation of the ellipsoid.

Consider a N-sensor dynamic linear varying system with unknown but bounded noises

$$x_k = F_{k-1}x_{k-1}+G_{k-1}w_{k-1} \tag{2}$$

$$z_{i,k}=H_{i,k}x_k+v_{i,k},\quad i=1,2,\cdots,N \tag{3}$$

where $x_k\in\mathbb{R}^n$ is the system state at time k, F_{k-1} and G_{k-1} are the state transition matrix and process noise input matrix, respectively. $z_{i,k}\in\mathbb{R}^{m_i}$ is the measurement of the i-th sensor, and $H_{i,k}$ is the corresponding observation matrix. $w_{k-1}\in\mathbb{R}^l$ and $v_{i,k}\in\mathbb{R}^{m_i}$ are process and observation noises assumed to be bounded by the following ellipsoids

$$\mathcal{W}_{k-1}=\{\boldsymbol{w}_{k-1}:w_{k-1}^T Q_{k-1}^{-1}w_{k-1}\le 1\} \tag{4}$$

$$\mathcal{V}_{i,k}=\{v_{i,k}:v_{i,k}R_{i,k}^{-1}v_{i,k}\le 1\} \tag{5}$$

where Q_{k-1} and $R_{i,k}$ are known matrices which are positive definite.

The initial state also lies in an ellipsoid given by

$$\mathcal{E}(\hat{x}_0,\sigma_0P_0)=\{x_0:(x_0-\hat{x}_0)^T P_0^{-1}(x_0-\hat{x}_0)\le\sigma_0\} \tag{6}$$

where $\sigma_0\in\mathbb{R}$ is a positive scalar variable.

For the distributed set-membership estimation fusion problem, the fusion center receives the local estimated state bounding ellipsoid $\mathcal{E}_{i,k}=\mathcal{E}(\hat{x}_{i,k},\sigma_{i,k}P_{i,k})$ at time k, which is obtained by local processors accompanying each sensor through prediction step and update step. Then the objective ellipsoid $\mathcal{E}_k$ will be calculated by using $\mathcal{E}_{i,k}$ in the fusion center and it must satisfy $\mathcal{E}_k\supseteq\bigcap_{i=1}^{N}\mathcal{E}_{i,k}$.

3. THE PROPOSED ALGORITHM

In this section, we design typical distributed set-membership fusion algorithms. The following lemma is the estimate for the local measurements with bounded disturbances.

Lemma 1 [13], [14]**:**

If $x_{k-1}\in\mathcal{E}_{i,k-1}=\mathcal{E}(\hat{x}_{i,k-1},\sigma_{i,k-1}P_{i,k-1})$ obeying (2) with $w_k\in\mathcal{E}(0,Q_k)$ and

$$\hat{x}_{i,k|k-1}=F_{k-1}\hat{x}_{i,k-1} \tag{7}$$

$$\sigma_{i,k|k-1}=\sigma_{i,k-1} \tag{8}$$

$$\begin{aligned}P_{i,k|k-1}&=(1+p_{i,k}^{-1})F_{k-1}P_{i,k-1}F_{k-1}^{\mathrm{T}}\\&+(1+p_{i,k})G_{k-1}Q_{k-1}G_{k-1}^{\mathrm{T}}/\sigma_{i,k|k-1}\end{aligned} \tag{9}$$

then

$\forall p_{i,k}\in(0,+\infty)$, $x_k\in\mathcal{E}_{i,k|k-1}=\mathcal{E}(\hat{x}_{i,k|k-1},\sigma_{i,k|k-1}P_{i,k|k-1})$.

If $x_k\in\mathcal{E}_{i,k|k-1}$ obeying (3) with $v_{i,k}\in\mathcal{E}(0,R_{i,k})$, $i\in\{1,2,\cdots N\}$

and

$$P_{i,k}^{-1}=P_{i,k|k-1}^{-1}+q_{i,k}H_{i,k}^{\mathrm{T}}R_{i,k}^{-1}H_{i,k} \tag{10}$$

$$\hat{x}_{i,k}=\hat{x}_{i,k|k-1}+q_{i,k}P_{i,k}H_{i,k}^{\mathrm{T}}R_{i,k}^{-1}\delta_{i,k} \tag{11}$$

$$\begin{aligned}\sigma_{i,k}&=q_{i,k}\left(1-\delta_{i,k}^{\mathrm{T}}R_{i,k}^{-1}\delta_{i,k}\right)\\&+\left(\hat{x}_{i,k}-\hat{x}_{i,k|k-1}\right)^{\mathrm{T}}P_{i,k}^{-1}\left(\hat{x}_{i,k}-\hat{x}_{i,k|k-1}\right)+\sigma_{i,k|k-1}\end{aligned} \tag{12}$$

where

$$\delta_{i,k}=z_{i,k}-H_{i,k}\hat{x}_{i,k|k-1} \tag{13}$$

then

$\forall q_{i,k}\in[0,+\infty)$, $x_k\in\mathcal{E}(\hat{x}_{i,k},\sigma_{i,k}P_{i,k})=\mathcal{E}_{i,k}\supseteq\mathcal{E}_{i,k|k-1}\cap\mathcal{X}_{i,k}$,

where $\mathcal{X}_{i,k}$ is given by

$$\mathcal{X}_{i,k}=\{x:(z_{i,k}-H_{i,k}x)^T R_{i,k}^{-1}(z_{i,k}-H_{i,k}x)\le 1\} \tag{14}$$

Remark 1: The optimal value of $p_{i,k}$ is chosen by minimizing $\sigma_{i,k|k-1}\mathrm{tr}P_{i,k|k-1}$ [24]. As for the parameter $q_{i,k}$, its optimal value is chosen based on Lemma 2 in [7]. This parameter selection scheme for $p_{i,k}$ and $q_{i,k}$ is also used in subsequent algorithms and theorems.

A. The global estimation at the fusion center

The fusion algorithm proposed by Becis-Aubry et al. [13], [14] is given below.

Algorithm 1.

If $x_{k-1}\in\mathcal{E}_{k-1}=\mathcal{E}(\hat{x}_{k-1},\sigma_{k-1}P_{k-1})$ obeying (2) with $w_k\in\mathcal{E}(0,Q_k)$, and given the local estimated ellipsoids $\mathcal{E}_{i,k}=\mathcal{E}(\hat{x}_{i,k},\sigma_{i,k}P_{i,k})$ and predicted ellipsoids $\mathcal{E}_{i,k|k-1}=\mathcal{E}(\hat{x}_{i,k|k-1},\sigma_{i,k|k-1}P_{i,k|k-1})$ obtained by Lemma 1, and

$$P_k^{-1}=P_{k|k-1}^{-1}+\sum_{i=1}^{N}\alpha_{i,k}(P_{i,k}^{-1}-P_{i,k|k-1}^{-1}) \tag{15}$$

$$\hat{x}_k=P_k\left[P_{k|k-1}^{-1}\hat{x}_{k|k-1}+\sum_{i=1}^{N}\alpha_{i,k}(P_{i,k}^{-1}\hat{x}_{i,k}-P_{i,k|k-1}^{-1}\hat{x}_{i,k|k-1})\right] \tag{16}$$

$$\sigma_k = \sigma_{k|k-1} + \left(\hat{x}_k - \hat{x}_{k|k-1}\right)^{\mathrm{T}} P_k^{-1} \left(\hat{x}_k - \hat{x}_{k|k-1}\right)$$
$$+\sum_{i=1}^{N} \alpha_{i,k} \left(\sigma_{i,k} - \sigma_{i,k|k-1} - \left(\hat{x}_{i,k} - \hat{x}_{i,k|k-1}\right)^{\mathrm{T}} P_{i,k}^{-1} \left(\hat{x}_{i,k} - \hat{x}_{i,k|k-1}\right)\right) \tag{17}$$

where $\hat{x}_{k|k-1}$, $P_{k|k-1}$ and $\sigma_{k|k-1}$ are global predicted estimates, and $\alpha_{i,k} \in [0,1], \sum_{i=1}^{N} \alpha_{i,k} = 1$,

then

$$x_k \in \mathcal{E}_k = \mathcal{E}\left(\hat{x}_k, \sigma_k P_k\right) \supseteq \mathcal{E}\left(\hat{x}_{k|k-1}, \sigma_{k|k-1} P_{k|k-1}\right) \cap \left(\bigcap_i \mathcal{X}_{i,k}\right)$$

where $\mathcal{X}_{i,k}$ is given by (14).

Remark 2: It shows that each local processor provides the following estimators

$$\left\{\hat{x}_{i,k}, P_{i,k}, \sigma_{i,k}, \hat{x}_{i,k|k-1}, P_{i,k|k-1}, \sigma_{i,k|k-1}\right\}$$

This means the global estimation requires not only local updates but also local predictions. And the central fusion processor also needs to perform the prediction process.

A simple substitution of the above feedback assignments in the equations of Algorithm 1 leads to another algorithm, as below.

Algorithm 2.

Given the local estimated ellipsoids $\mathcal{E}_{i,k} = \mathcal{E}\left(\hat{x}_{i,k}, \sigma_{i,k} P_{i,k}\right)$ obtained by Lemma 1 with $x_k \in \mathcal{E}_{i,k}$, $i \in \{1, 2, \cdots N\}$, then the global estimate ellipsoid $\mathcal{E}_k = \mathcal{E}\left(\hat{x}_k, \sigma_k P_k\right)$ with

$$P_k^{-1} = \sum_{i=1}^{N} \alpha_{i,k} P_{i,k}^{-1} \tag{18}$$

$$\hat{x}_k = P_k \sum_{i=1}^{N} \alpha_{i,k} P_{i,k}^{-1} \hat{x}_{i,k} \tag{19}$$

$$\sigma_k = \sum_{i=1}^{N} \alpha_{i,k} \sigma_{i,k} - \sum_{i=1}^{N} \alpha_{i,k} \hat{x}_{i,k}^{\mathrm{T}} P_{i,k}^{-1} \hat{x}_{i,k} + \hat{x}_k^{\mathrm{T}} P_k^{-1} \hat{x}_k \tag{20}$$

satisfies $\mathcal{E}_k \supseteq \bigcap_{i=1}^{N} \mathcal{E}_{i,k}$,

where $\alpha_{i,k} \in [0,1], \sum_{i=1}^{N} \alpha_{i,k} = 1$.

Proof.

For Algorithm 1, set $\hat{x}_{i,k-1} \leftarrow \hat{x}_{k-1}$, $P_{i,k-1} \leftarrow P_{k-1}$, $\sigma_{i,k-1} \leftarrow \sigma_{k-1}$,

then it is obvious that $\hat{x}_{i,k|k-1} = \hat{x}_{k|k-1}$, $P_{i,k|k-1} = P_{k|k-1}$, and $\sigma_{i,k|k-1} = \sigma_{k|k-1}$. Substituting this into (15), (16) and (17), the following results can be deduced.

$$P_k^{-1} = P_{k|k-1}^{-1} + \sum_{i=1}^{N} \alpha_{i,k} (P_{i,k}^{-1} - P_{k|k-1}^{-1}) = \sum_{i=1}^{N} \alpha_{i,k} P_{i,k}^{-1} \tag{21}$$

$$\hat{x}_k = P_k \left(P_{k|k-1}^{-1} \hat{x}_{k|k-1} + \sum_{i=1}^{N} \alpha_{i,k} (P_{i,k}^{-1} \hat{x}_{i,k} - P_{k|k-1}^{-1} \hat{x}_{k|k-1}) \right)$$
$$= P_k \sum_{i=1}^{N} \alpha_{i,k} P_{i,k}^{-1} \hat{x}_{i,k} \tag{22}$$

$$\sigma_k = \left(\hat{x}_k - \hat{x}_{k|k-1}\right)^{\mathrm{T}} P_k^{-1} \left(\hat{x}_k - \hat{x}_{k|k-1}\right)$$
$$+\sum_{i=1}^{N} \alpha_{i,k} \left[\sigma_{i,k} - \left(\hat{x}_{i,k} - \hat{x}_{k|k-1}\right)^{\mathrm{T}} P_{i,k}^{-1} \left(\hat{x}_{i,k} - \hat{x}_{k|k-1}\right)\right]$$
$$= \sum_{i=1}^{N} \alpha_{i,k} \sigma_{i,k} - \sum_{i=1}^{N} \alpha_{i,k} \hat{x}_{i,k}^{\mathrm{T}} P_{i,k}^{-1} \hat{x}_{i,k} + \left(\hat{x}_k - \hat{x}_{k|k-1}\right)^{\mathrm{T}} P_k^{-1} \left(\hat{x}_k - \hat{x}_{k|k-1}\right)$$
$$+2\hat{x}_{k|k-1}^{\mathrm{T}} P_k^{-1} \hat{x}_k - \hat{x}_{k|k-1}^{\mathrm{T}} P_k^{-1} \hat{x}_{k|k-1}$$
$$= \sum_{i=1}^{N} \alpha_{i,k} \sigma_{i,k} - \sum_{i=1}^{N} \alpha_{i,k} \hat{x}_{i,k}^{\mathrm{T}} P_{i,k}^{-1} \hat{x}_{i,k} + \hat{x}_k^{\mathrm{T}} P_k^{-1} \hat{x}_k \tag{23}$$

The proof is completed.

Remark 3: In this algorithm, the global estimate is obtained based only on the local estimated ellipsoids, which means that it requires less bandwidth than Algorithm 1 to transfer data between the local processor and the central processor. And the central fusion processor does not need to perform the prediction process, which means a reduction of computational burden.

In addition, if the local processors operate in an open-loop manner, they cannot correct the estimate deviation caused by their own output error or estimate error, which ultimately leads to the local track deviation and further affects the overall estimate. A feedback from central processor to local processor can improve that situation. In the algorithm with feedback, the fused data at previous time ($\hat{x}_{k-1}$, P_{k-1} and σ_{k-1}) are returned to the local processors and used instead of the local estimates at previous time ($\hat{x}_{i,k-1}$, $P_{i,k-1}$ and $\sigma_{i,k-1}$).

B. *Selection scheme of the parameters*

As the parameter for a family of ellipsoids contains the intersection of multiple ellipsoids, $\alpha_{i,k}$ is usually chosen by minimizing the determinant or trace of the matrix P_k [22]. But this solution is not retained in this paper for two reasons. Firstly, the solution cannot yield an explicit solution in general and thus the convex optimization problem arises. The heavy computation burden restricts its application. Secondly, this algorithm should reject the outliers of the measurements. The outlier may make the local estimated ellipsoid inconsistent with the others. $\alpha_{i,k}$ should not only guarantee the robust feasible set, but also reduce the effect of the outliers on the global estimate. Then $\alpha_{i,k}$ is computed considering the following assumption:

The q-relaxed intersection of m sets $\mathcal{X}_1, \mathcal{X}_2, \cdots, \mathcal{X}_m$ is denoted by $\mathcal{X}^{\{q\}} = \bigcap^{\{q\}} \mathcal{X}_i$, which is the set containing all x belonging to all $\mathcal{X}_i$'s except q at most. And in the distributed fusion problem, we focus on the local estimated sets $\mathcal{E}_{i,k}$. At time k, the midpoint of the set $\mathcal{E}_k^{\{q\}} = \overset{\{q\}}{\bigcap} \mathcal{E}_{i,k}$, denoted by $\hat{x}_{k,mp}$, can be used to calculate the parameter $\alpha_{i,k}$. To reduce the effect of outliers, $\alpha_{i,k}$ is chosen as a decreasing function of some norm of the difference between the midpoint $\hat{x}_{k,mp}$ and the center of the local estimated ellipsoid $\mathcal{E}_{i,k}$, denoted as

$$\left\|\Delta x_{i,k}\right\|_{P_{i,k}^{-1}} \triangleq \left(\hat{x}_{i,k} - \hat{x}_{k,mp}\right)^{\mathrm{T}} P_{i,k}^{-1} \left(\hat{x}_{i,k} - \hat{x}_{k,mp}\right)$$

Then the parameter $\alpha_{i,k}$ is chosen as

$$\alpha_{i,k} = \frac{g\left(\left\|\Delta x_{i,k}\right\|_{P_{i,k}^{-1}}\right)}{\sum_{j=1}^{N} g\left(\left\|\Delta x_{i,k}\right\|_{P_{i,k}^{-1}}\right)} \tag{23}$$

where $g\left(\left\|\Delta x_{i,k}\right\|_{P_{i,k}^{-1}}\right) = \begin{cases} 1 - \left\|\Delta x_{i,k}\right\|_{P_{i,k}^{-1}}, if \ \hat{x}_{k,mp} \in \mathcal{E}_{i,k} \\ 0 \qquad\qquad\qquad , otherwise \end{cases}$

It is difficult to compute an ellipsoid enclosing the q-relaxed intersection of *m* ellipsoids. Alternatively, we use the minimal box containing the ellipsoid to compute the relaxed midpoint $\hat{x}_{k,rmp}$, as a replacement of $\hat{x}_{k,mp}$ in the calculation of $\alpha_{i,k}$.

First, the extrema of the local ellipsoid $\mathcal{E}_{i,k}$ are found as

$$\hat{x}_{i,k,\pm}^{j} = \hat{x}_{i,k}^{j} \pm \sqrt{P_{i,k}^{j,j}}$$

where the superscript j denotes the j-th state and the subscripts + and - denote the maximum and minimum values, respectively. The box or the interval vector $X_{i,k}$ is then defined as

$$X_{i,k}^{j} = [\hat{x}_{i,k,-}^{j}, \hat{x}_{i,k,+}^{j}]$$

Now a box enclosing the q-relaxed intersection of *m* boxes is computed by an algorithm that is presented in [23].

Algorithm 1 is actually a set-membership distributed fusion (SMDF) method proposed by Becis-Aubry. For convenience, it is abbreviated as BA-SMDF in this paper. In Algorithm 2, the q-relaxed intersection of ellipsoids is used in the choice of the weighting parameters. Thus, Algorithm 2 is abbreviated as QSMDF. The two algorithms with feedback are abbreviated as FBA-SMDF and FQSMDF, respectively.

4. Equivalence analysis

In this section, the equivalence of the above algorithms is analyzed, and following conclusions are derived.

Theorem 1. Considering a N-sensor system given by (2) and (3), FBA-SMDF algorithm and FQSMDF algorithm are functionally equivalent in terms of both global and local estimation accuracy if the parameters of the two algorithms are chosen to be identical.

Proof.

Obviously, for these two algorithms, it is reasonable for the global and local trackers to have the same initial value. Assume these two algorithms have the same estimate at time k-1. Then Theorem 1 is obvious according to the proof process of Algorithm 2.

Theorem 2. Considering a N-sensor system given by (2) and (3), FBA-SMDF algorithm and BA-SMDF algorithm are functionally equivalent in terms of global estimation accuracy if the parameters of the two algorithms are chosen to be identical.

Proof.

In order to distinguish the case without feedback, $\hat{\hat{x}}_{i,k|k-1}$, $\hat{\hat{x}}_{k|k-1}$, $\hat{\hat{x}}_{i,k}$, $\hat{\hat{x}}_{k}$, $\hat{P}_{i,k|k-1}$, $\hat{P}_{k|k-1}$, $\hat{P}_{i,k}$, $\hat{P}_{k}$, $\hat{\sigma}_{i,k|k-1}$, $\hat{\sigma}_{k|k-1}$, $\hat{\sigma}_{i,k}$ and $\hat{\sigma}_{k}$ is used to describe the results of FBA-SMDF algorithm.

Obviously, with or without feedback, it is reasonable for the global and local trackers to have the same initial value, i.e.

$$\hat{\hat{x}}_0 = \hat{x}_0 = \hat{\hat{x}}_{i,0} = \hat{x}_{i,0}, \quad \hat{P}_0 = P_0 = \hat{P}_{i,0} = P_{i,0},$$

$$\hat{\sigma}_0 = \sigma_0 = \hat{\sigma}_{i,0} = \sigma_{i,0}$$

Assume $\hat{\hat{x}}_{k-1} = \hat{x}_{k-1}$, $\hat{P}_{k-1} = P_{k-1}$, $\hat{\sigma}_{k-1} = \sigma_{k-1}$,

and it is apparent that

$$\hat{\hat{x}}_{k|k-1} = \hat{x}_{k|k-1}, \quad \hat{P}_{k|k-1} = P_{k|k-1}, \quad \hat{\sigma}_{k|k-1} = \sigma_{k|k-1}.$$

Let $\hat{\hat{x}}_{i,k-1} \leftarrow \hat{\hat{x}}_{k-1}$, $\hat{P}_{i,k-1} \leftarrow \hat{P}_{k-1}$, $\hat{\sigma}_{i,k-1} \leftarrow \hat{\sigma}_{k-1}$,

then we have

$$\hat{\hat{x}}_{i,k|k-1} = \hat{x}_{k|k-1}, \quad \hat{P}_{i,k|k-1} = P_{k|k-1}, \quad \hat{\sigma}_{i,k|k-1} = \sigma_{k|k-1}$$

and

$$\begin{aligned} \hat{P}_k^{-1} &= \sum_{i=1}^{N} \alpha_{i,k} \hat{P}_{i,k}^{-1} = \sum_{i=1}^{N} \alpha_{i,k} \left(\hat{P}_{i,k|k-1}^{-1} + q_{i,k} H_{i,k}^{\mathrm{T}} R_{i,k}^{-1} H_{i,k}\right) \\ &= P_{k|k-1}^{-1} + \sum_{i=1}^{N} \alpha_{i,k} (P_{i,k}^{-1} - P_{i,k|k-1}^{-1}) = P_k^{-1} \end{aligned} \tag{24}$$

$$\begin{aligned} \hat{\hat{x}}_k &= \hat{P}_k^{-1} \left(\sum_{i=1}^{N} \alpha_{i,k} \hat{P}_{i,k}^{-1} \hat{\hat{x}}_{i,k}\right) \\ &= \hat{P}_k^{-1} \left(\sum_{i=1}^{N} \alpha_{i,k} \left(\hat{P}_{i,k|k-1}^{-1} \hat{\hat{x}}_{i,k|k-1} + q_{i,k} H_{i,k}^{\mathrm{T}} R_{i,k}^{-1} z_{i,k}\right)\right) \\ &= P_k^{-1} \left(P_{k|k-1}^{-1} \hat{x}_{k|k-1} + \sum_{i=1}^{N} \alpha_{i,k} \left(P_{i,k}^{-1} \hat{x}_{i,k} - P_{i,k|k-1}^{-1} \hat{x}_{i,k|k-1}\right)\right) \\ &= \hat{x}_k \end{aligned} \tag{25}$$

$$\begin{aligned}\hat{\sigma}_k &= \left(\hat{\hat{x}}_k - \hat{\hat{x}}_{k|k-1}\right)^{\mathrm{T}} \hat{P}_k^{-1}\left(\hat{\hat{x}}_k - \hat{\hat{x}}_{k|k-1}\right) \\ &+\sum_{i=1}^{N}\alpha_{i,k}\left(\hat{\sigma}_{i,k} - \left(\hat{\hat{x}}_{i,k} - \hat{\hat{x}}_{k|k-1}\right)^{\mathrm{T}} \hat{P}_{i,k}^{-1}\left(\hat{\hat{x}}_{i,k} - \hat{\hat{x}}_{k|k-1}\right)\right) \\ &= \left(\hat{\hat{x}}_k - \hat{\hat{x}}_{k|k-1}\right)^{\mathrm{T}} \hat{P}_k^{-1}\left(\hat{\hat{x}}_k - \hat{\hat{x}}_{k|k-1}\right) \\ &+\sum_{i=1}^{N}\alpha_{i,k}\left(q_k\left(1-\hat{\delta}_{i,k}^{\mathrm{T}} R_{i,k}^{-1}\hat{\delta}_{i,k}\right)+\hat{\sigma}_{i,k|k-1}\right) \\ &= \sigma_{k|k-1} + \left(\hat{x}_k - \hat{x}_{k|k-1}\right)^{\mathrm{T}} P_k^{-1}\left(\hat{x}_k - \hat{x}_{k|k-1}\right) \\ &+\sum_{i=1}^{N}\alpha_{i,k}q_{i,k}\left(1-\delta_{i,k}^{\mathrm{T}} R_{i,k}^{-1}\delta_{i,k}\right) = \sigma_k\end{aligned} \tag{26}$$

The proof is completed.

Remark 4: It is revealed in Theorem 2 that the feedback is not needed to improve the global estimation accuracy of BA-SMDF. It only impacts the local estimates and since the best estimation possible of the state vector is reasonably produced by the global process, this feedback can be considered superfluous.

5. SIMULATIONS

Some simulations are performed to assess the algorithm performance and verify the corresponding conclusions.

Consider the target tracking system formulated in (2) and (3) with 3 sensors, and the required matrices are given as

$$F_k = \begin{bmatrix} 1 & T_0 & 0.5T_0^2 \\ 0 & 1 & T_0 \\ 0 & 0 & 1 \end{bmatrix}, G_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$H_{1,k} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, H_{2,k} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, H_{3,k} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},$$

where $T_0 = 0.1$ is the sampling period, and the state is expressed as $x_k = \begin{bmatrix} x_{1,k} & x_{2,k} & x_{3,k} \end{bmatrix}^{\mathrm{T}}$. The matrices in (4) and (5) are given as $Q_k = \mathrm{diag}(10,10,10)$, $R_{1,k} = \mathrm{diag}(0.2,0.2)$, $R_{2,k} = \mathrm{diag}(0.8,0.6)$, $R_{3,k} = 0.7$. Besides, the parameters of the initial state are given as $P_0 = 100I_3$, $\hat{x}_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$, $\sigma_0 = 1$.

In the simulation, the process noises and observation noises are uniformly distributed inside the ellipsoids, as illustrated in Fig.1. and Fig.2. Additionally, the observation noises of sensor 3 are uniformly distributed in the interval $\left[-\sqrt{0.7}, \sqrt{0.7}\right]$, which can be seen as an ellipsoid with one dimension.

Two cases are considered for simulation:

Case 1: In the first scenario, to verify the correctness of the conclusions in Section 5, all algorithms are performed with $\alpha_{i,k} = \dfrac{1}{3}$.

Case 2: In the second scenario, for QSMDF and FQSMDF, the parameters are calculated according to (24). For FBA-SMDF and BA-SMDF, the parameters are calculated according to [13]-[14].

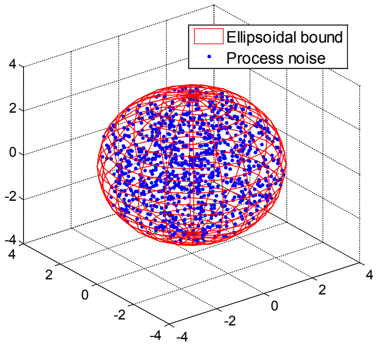
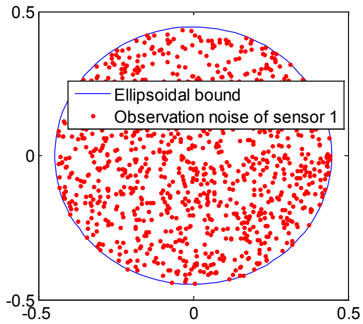
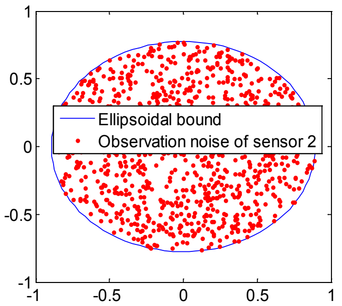


Fig.1. The distribution of process noise.



a)



b)

Fig.2. The distribution of observation noise: a) sensor 1; b) sensor 2.

The simulations are run 100 times under Matlab R2019a on Intel Core i5 PC (3.2 GHz, 4G RAM) and each simulation contains 1000 samples. The average mean square error (MSE) in each state variable of the global state estimates, the average volume of the estimated ellipsoid over 100 simulations are used as the evaluation indices, which are illustrated in Table 1. and Table 2. It should be noted that the MSE and volume are averaged from step 100 to step 1000 to rule out the influence of the initial phase and the center of the ellipsoid is considered to be the point estimate in the simulation. Furthermore, the MSE and the volume overtime in one simulation (Case 1) are shown in Fig.3. and Fig.4. to display the results more visually.

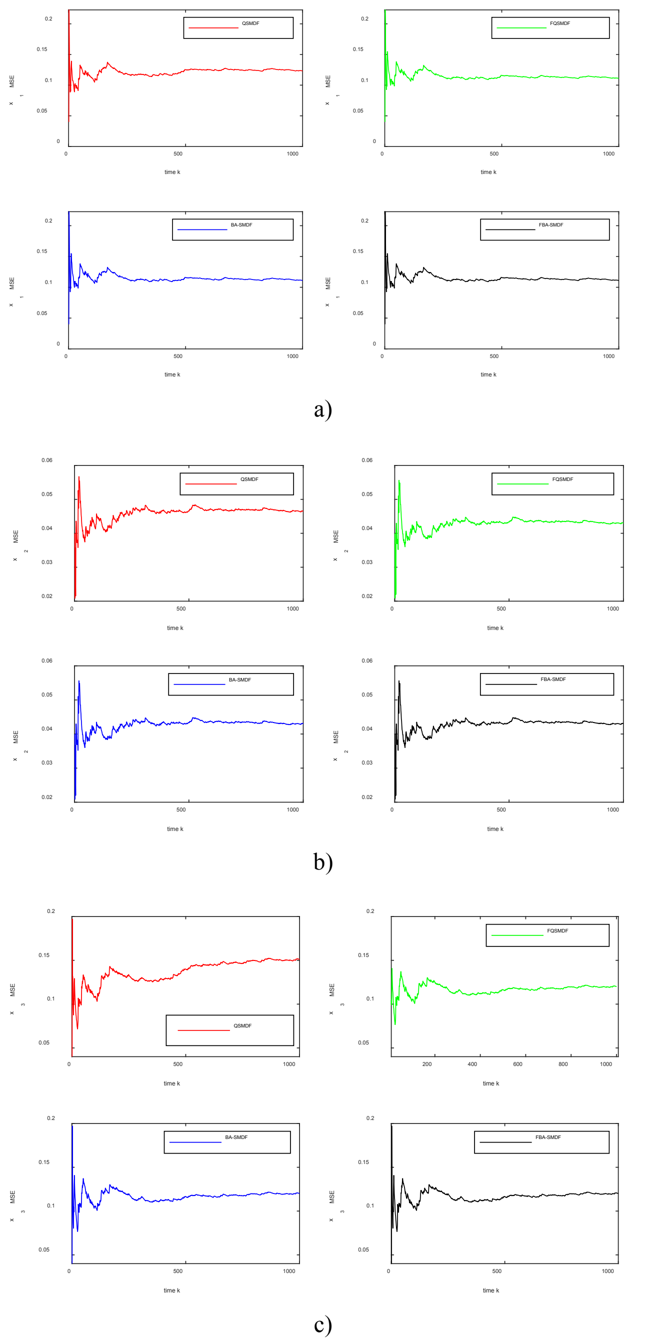


Fig.3. MSE of states for overall estimates. a) MSE for states $i=1$; b) MSE for states $i=2$.; c) MSE for states $i=3$.

The results in Table 1. show that the MSE and the volume of the FQSMDF, FBA-SMDF and BA-SMDF are equivalent in Case 1. This verifies the correctness of the theorems in Section 4. The same conclusion can also be obtained from the figures.

From Table 2., it can be found that after optimizing the parameters, the estimation accuracy (in terms of MSE) of each algorithm is improved. Especially, the FQSMDF algorithm proposed in this paper has the highest accuracy among the above algorithms. It should be noted that the volume of the ellipsoid does not decrease significantly because the parameter is not chosen by minimizing the volume.

Table 1. The average MSE and volume of bounding ellipsoids for overall estimates (Case 1).

Algorithms	MSE			Volume
	$\overline{e}_1^2$ [1]	$\overline{e}_2^2$	$\overline{e}_3^2$	
QSMDF	0.1167	0.0472	0.1468	2.5612e+03
FQSMDF	0.1084	0.0437	0.1180	85.8558
BA-SMDF	0.1084	0.0437	0.1180	85.8558
FBA-SMDF	0.1084	0.0437	0.1180	85.8558

[1] $\overline{e}_i^2$ refers to the average MSE, for states $i=1,2,3$.

Table 2. The average MSE and volume of bounding ellipsoids for overall estimates (Case 2).

Algorithms	MSE			Volume
	$\overline{e}_1^2$	$\overline{e}_2^2$	$\overline{e}_3^2$	
QSMDF	0.1076	0.0412	0.1286	2.352e+03
FQSMDF	0.0652	0.0286	0.0776	84.3244
BA-SMDF	0.0724	0.0303	0.0847	84.4354
FBA-SMDF	0.0724	0.0303	0.0847	84.4354

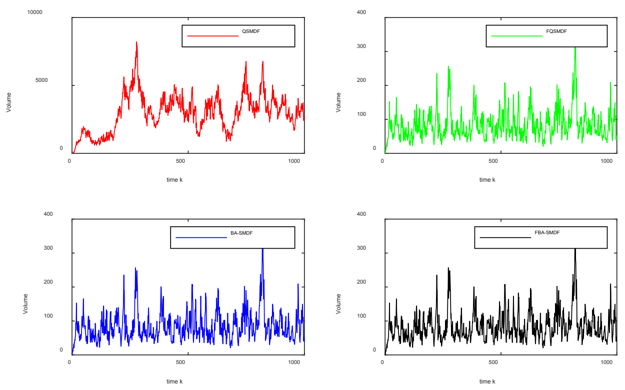


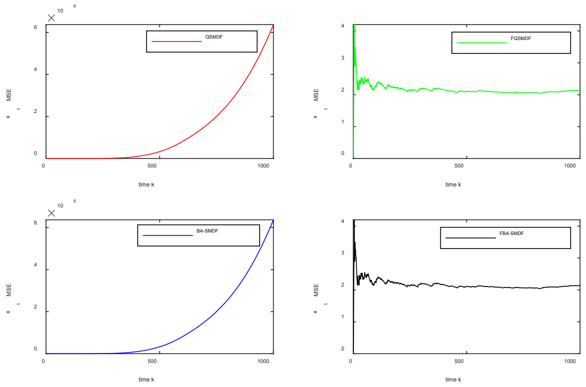
Fig.4. The volume of overall estimates of estimated bounding ellipsoids.

In addition, average MSE and volume of bounding ellipsoids for local estimates are also calculated, as shown in Table 3.（Case 1）. Especially, Fig.5. and Fig.6. show the MSE and the volume of the estimated ellipsoids for sensor 3 over time in one simulation. It can be concluded from Table 3. that for the set-membership distributed fusion algorithms in this paper, feedback can significantly improve the local estimation accuracy of each sensor. Moreover, the estimation using the algorithms with feedback has better convergence than those without feedback, as illustrated in Fig.5 and Fig.6. And it can be seen from Table 3. and Fig.5. And Fig.6. that the local estimates of the FQSMDF are equal to those of FBA-SMDF and the local estimates of the QSMDF are equal to those of BA-SMDF. Combined with the conclusions from Table 1., Theorem 1 is validated. Then contrast of Table 3. and Table 1. shows that the global estimation has higher precision than the local estimation for each variable, which fully illustrates the effectiveness of the fusion.

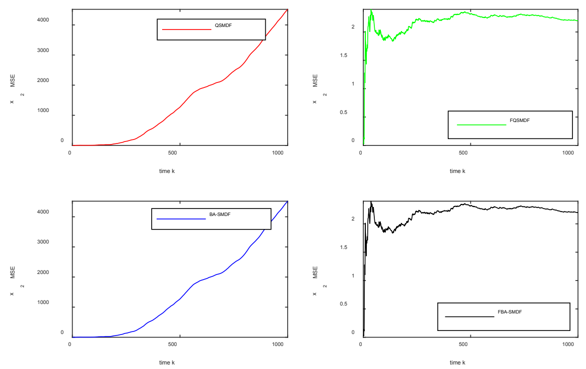
In terms of computational time, there is no obvious difference between the above algorithms. The average computational time at each recursive step is about 0.15 ms, which meets the needs of real-time application.

Table 3. The average MSE and volume of bounding ellipsoids for local estimates.

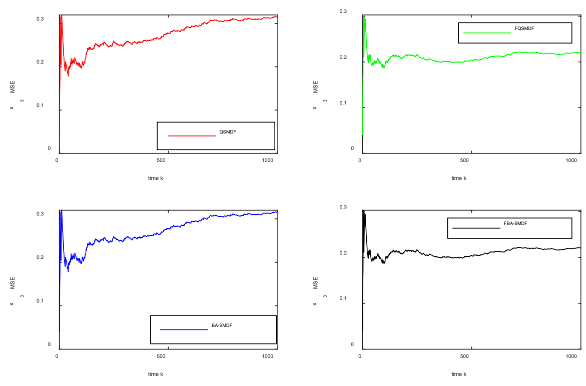
Sensor	Algorithms	MSE			Volume
		$\bar{e}_1^2$	$\bar{e}_2^2$	$\bar{e}_3^2$	
1	QSMDF BA-SMDF	41.3166	0.0495	41.3150	1.2423e+04
	FQSMDF FBA-SMDF	1.1173	0.0488	1.1160	95.9862
2	QSMDF BA-SMDF	0.1583	0.1936	43.2704	2.1940e+05
	FQSMDF FBA-SMDF	0.1402	0.1845	2.1295	1.7566e+03
3	QSMDF BA-SMDF	1.6792e+06	1.0328e+03	0.3009	Inf
	FQSMDF FBA-SMDF	2.1211	2.0428	0.2121	2.1766e+04



a)



b)



c)

Fig.5. The average MSE for local estimates of Sensor 3. a) MSE for states $i=1$; b) MSE for states $i=2$; c) MSE for states $i=3$.

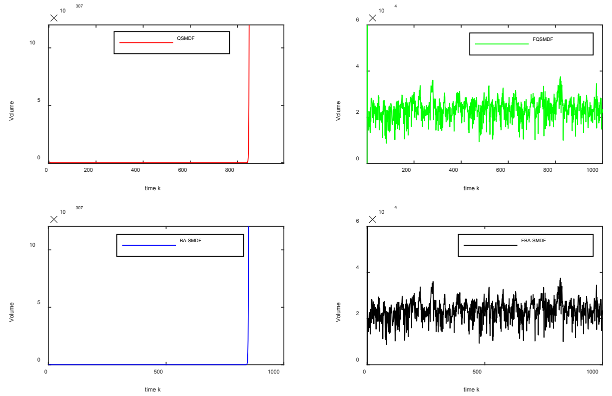


Fig.6. The volume of bounding ellipsoids for local estimates of Sensor 3.

6. CONCLUSIONS

Based on the set-membership theory, an outer-bounding state fusion estimation algorithm with distributed structure has been proposed. Theoretical and simulation results on the comparison of the proposed algorithm (QSMDF) and BA-SMDF algorithm are also presented, including the algorithms with feedback and without feedback. Conclusions are summarized as below:

1. The proposed algorithm with feedback has higher accuracy than BA-SMDF, largely due to its selection scheme of the parameters.
2. The proposed algorithm requires less bandwidth than BA-SMDF to transfer data between the local processor and the central processor.
3. The two algorithms with feedback are functionally equivalent in terms of both global and local estimation accuracy if the parameters of the two algorithms are chosen to be identical.
4. For QSMDF algorithm, feedback can improve both global and local estimation accuracy.
5. For BA-SMDF algorithm, feedback can improve the local estimation accuracy, but cannot improve the global estimation accuracy further.

In addition, the effectiveness of the proposed algorithm in improving state estimation accuracy is also proved by the simulation results.

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